Anotes10
Calculus Theory and Limits

Theorem 10.1. (The Maximum value theorem.) If f is a function whose domain contains the interval $[a, b]$ so that f is continuous at each point with x-value in the interval, then there is a maximum value for f on the interval $[a, b]$.

Theorem 10.2 . (Intermediate value theorem.) If f is continuous on the interval $[a, b]$ and c is a number so that c is between $f(a)$ and $f(b)$ then there is a number $t_0 \in (a, b)$ so that $f(t_0) = c$.

Definition. Let $f : D \to \mathbb{R}$ be a function. Then

$$\lim_{x \to a} f(x) = L$$

means that a is a limit point of D and

a. if U is an open set containing L then there is an open set V containing a so that $f((V - \{a\}) \cap D) \subset U$.

b. if $\epsilon > 0$ there exists a number δ_ϵ so that if $x \in D$ and $0 < |x - a| < \delta_\epsilon$ then $|f(x) - L| < \epsilon$.

Definition. Let $f : D \to \mathbb{R}$ be a function. Then

$$\lim_{x \to \infty} f(x) = L$$

means that D does not have an upper bound and

a. if U is an open set containing L then there is a number B so that $f(\{x | x > B\} \cap D) \subset U$.

b. if $\epsilon > 0$ there exists a number B so that if $x \in D$ and $x > B$ then $|f(x) - L| < \epsilon$.

Exercise: 10.1. Explain the relationship between continuity and limits.

Theorem 10.3. [Theorems about limits.] Suppose that f and g are functions, with a common domain, whose limits exist at the point $x = a$ with:

$$\lim_{x \to a} f(x) = A \text{ and } \lim_{x \to a} g(x) = B.$$

Assume that C is a constant. Then:

$$
\begin{array}{ll}
a. & \lim_{x \to a} f(x) + g(x) = A + B \\
b. & \lim_{x \to a} Cf(x) = CA \\
c. & \lim_{x \to a} f(x) \cdot g(x) = A \cdot B \\
d. & \text{if } A \neq 0, \\
 & \text{then } \lim_{x \to a} \dfrac{1}{f(x)} = \dfrac{1}{A}.
\end{array}
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