Axioms of the integers $\mathbb{Z}$.

The word "integer" is our undefined term (sometimes referenced as the "primitive" term) for this section. The set $\mathbb{Z}$ is the set of all integers (Axiom D3 implies that $\mathbb{Z}$ has at least two elements, so I am grammatically correct in using the plural). The set $\mathbb{Z}$ satisfies the following axioms.

The usual rules (axioms) of logic are to be used to prove theorems from these axioms. As needed these rules will be discussed and stated. As a first such, following are the properties of the equality symbol.

i.) [Reflexive] $x = x$ for all x.
ii.) [Symmetric] If $x = y$ then $y = x$.
iii.) [Transitive] If $x = y$ and $y = z$ then $x = z$.
iv.) [Uniqueness of function values] If f is a function and $x = y$ then $f(x) = f(y)$.

Axioms about addition and multiplication: There exists two operations on the integers: addition denoted by "+" and multiplication denoted by "·". [Strictly speaking "+" is a map from the cross product of the integers with itself into the integers with certain properties as defined by the axioms, and similarly for multiplication "·".] Note that $a \cdot b$ is usually written as ab. It is this functional definition and properties of the = symbol that yields the following (axioms) which will be needed for proofs and for which you may assume as part of our logic system.

If $a = b$ and $c = d$ then $a + c = b + d$.
If $a = b$ and $c = d$ then $a \cdot c = b \cdot d$.
[Observation: the symbols $\wedge$ and $\vee$ also satisfy these properties.]

Axioms about addition.

A1. If $a \in \mathbb{Z}$ and $b \in \mathbb{Z}$ then $a + b \in \mathbb{Z}$. [Closure.]

A2. If $a \in \mathbb{Z}$ and $b \in \mathbb{Z}$ then $a + b = b + a$. [Commutativity.]

A3. If $a \in \mathbb{Z}$, $b \in \mathbb{Z}$ and $c \in \mathbb{Z}$ then $a+(b+c) = (a+b)+c$. [Associativity.]

A4. There exists an element $0 \in \mathbb{Z}$ so that if $a \in \mathbb{Z}$ then $a + 0 = a$. [Additive Identity element.]

A5. If $a \in$ then there exits an element in $\mathbb{Z}$ denoted by $-a$ so that $-a + a = 0$. [Additive inverse.]

Definition: $a - b$ means $a + (-b)$.

Axioms about multiplication.

B1. If $a \in \mathbb{Z}$ and $b \in \mathbb{Z}$ then $a \cdot b \in \mathbb{Z}$. [Closure.]

B2. If $a \in \mathbb{Z}$ and $b \in \mathbb{Z}$ then $a \cdot b = b \cdot a$. [Commutativity.]

B3. If $a \in \mathbb{Z}$, $b \in \mathbb{Z}$ and $c \in \mathbb{Z}$ then $a \cdot (b \cdot c) = (a \cdot b) \cdot c$. [Associativity.]

B4. There exists an element $1 \in \mathbb{Z}$ so that if $a \in \mathbb{Z}$ then $a \cdot 1 = a$. [Multiplicative Identity element.]

B5. If $a \in \mathbb{Z}$, $b \in \mathbb{Z}$, $c \in \mathbb{Z}$ with $c \neq 0$ and $ac = bc$ then $a = b$. [Cancellation rule.]

Axiom on the relationship between addition and multiplication.

C1. If $a \in \mathbb{Z}$, $b \in \mathbb{Z}$ and $c \in \mathbb{Z}$ then $a \cdot (b + c) = a \cdot b + a \cdot c$. [Distributive law.] (Note the assumption that the order of operation is to perform $\cdot$ first then $+$. In other words $ab + cd$ means $(ab) + (cd)$.)

Axioms on order.

There exists an order relation "$<$" so that:

D1. If each of a and b is an integer then exactly one of the following is true:

$$a < b, \quad a = b, \quad b < a.$$

D2. If each of a, b and c is an integer $a < b$ and $b < c$, then $a < c$.

D3. $0 < 1$.

D4. If each of a, b and c is an integer and $a < b$, then $a + c < b + c$.

D5. If each of a, b and c is an integer $a < b$ and $0 < c$, then $a \cdot c < b \cdot c$.

Definition. The integer p is said to be positive if and only if $0 < p$; $\mathbb{N} = \{n \in \mathbb{Z} | 0 < n\}$.

D6. If n is an integer then exactly one of the following is true [Trichotomy Law.]:

$$n \in \mathbb{N}, \quad n = 0, \quad -n \in \mathbb{N}.$$

Notation: The statement $a > b$ means that $b < a$.

Induction axiom.

E1. Suppose that S is a subset of $\mathbb{N}$ containing 1 such that if $n \in S$ then $n + 1 \in S$. Then $S = \mathbb{N}$.

An alternate, but equivalent, statement of the induction axiom is the following:
E1'. If S is a non-empty subset of $\mathbb{N}$, then S has a least element.

Strictly speaking, the positive integers $\mathbb{N}$ is defined inductively as follows:

$$1 \quad \in \quad \mathbb{N} \tag{1}$$
$$\text{if } n \in \mathbb{N} \quad \text{then} \quad n + 1 \in \mathbb{N} \tag{2}$$

and $\mathbb{N}$ is the minimal set satisfying conditions (1) and (2) above.
Then axiom E1 tells us that anything satisfying conditions (1) and (2) is $\mathbb{N}$.