Axioms of the real numbers $\mathbb{R}$.

Axioms on addition:

Axiom A1 (closure property of addition). If each of x and y is a number then $x+y$ is a number.

Axiom A2 (associative property of addition). If each of x, y and z is a number then $(x+y)+z = x+(y+z)$.

Axiom A3 (commutative property of addition). If each of x and y is a number then $x+y = y+x$.

Axiom A4 (identity for addition). There exists a number 0 so that if x is a number then $0+x = x$.

Axiom A5 (inverses for addition). If x is a number, there is a number denoted by $-x$ so that $x+(-x) = 0$.

Axioms on multiplication:

Axiom M1 (closure property of multiplication). If each of x and y is a number then $x \cdot y$ is a number.

Axiom M2 (associative property of multiplication). If each of x, y and z is a number then $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.

Axiom M3 (commutative property of multiplication). If each of x and y is a number then $x \cdot y = y \cdot x$.

Axiom M4 (identity for multiplication). There exists a number 1 so that if x is a number then $1 \cdot x = x$.

Axiom M5. $0 \neq 1$.

Axiom M6 (inverses for multiplication). If x is a number and $x \neq 0$, there is a number denoted by $\frac{1}{x}$ so that $x \cdot \frac{1}{x} = 1$.

Relationship between addition and multiplication:

Axiom D (distributive axiom). If each of x, y and z is a number then $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$.

Order axioms.

There exists an order relation "$<$" so that:

Axiom O1. If each of x and y is a number then exactly one of the following is true:

$$x < y, \qquad x = y, \qquad \text{or } y < x.$$

Axiom O2. If each of x, y and z is a number $x < y$ and $y < z$, then $x < z$.

Axiom O3. $0 < 1$.

Axiom O4. If each of x, y and z is a number and $x < y$, then $x+z < y+z$.

Axiom O5. If each of x, y and z is a number $x < y$ and $0 < z$, then $x \cdot z < y \cdot z$.

Definition. The integer p is said to be positive if and only if $0 < p$; $\mathbb{N} = \{n \in \mathbb{Z} | 0 < n\}$.

Induction Axiom. Note: We need to establish the relationship between the integers and the reals. To do that we define 0 and 1 to be integers and inductively (using the induction axiom) to define the positive integers $\mathbb{N}$ by: if $n \in \mathbb{N}$ then $n + 1 \in \mathbb{N}$. Then the integers $\mathbb{Z}$ consists of 0 together with $\mathbb{N}$ together with the additive inverses of elements of $\mathbb{N}$.

Axiom I. Suppose that S is a subset of $\mathbb{N}$ containing 1 such that if $n \in S$ then $n + 1 \in S$. Then $S = \mathbb{N}$.

Theorem. The induction axiom is equivalent to the following: Every non-empty subset of $\mathbb{N}$ has a least element.

Completeness Axiom. Definition: if M is a subset of the real numbers then B is an upper bound for M means that $x \leq B$ for all $x \in M$; B is said to a least upper bound of M if B is an upper bound of M and no element less than B is an upper bound of M.

Axiom C. If M is a subset of the reals $\mathbb{R}$ and there is an upper bound of M then there is a least upper bound of M.

Comments:

The addition axioms (A1 - A5) hold for $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$.

The multiplication axiom M1 - M6 hold for $\mathbb{Q}, \mathbb{R}$ but not for $\mathbb{Z}$ because to get multiplicative inverses you need to extend the integers to at least the rationals $\mathbb{Q}$.

The rationals satisfy all the axioms except the completeness axiom. You need the completeness axiom to prove that things like $\sqrt{2}$ exist. If we have time I'll outline a proof later in the semester. The complex numbers $\mathbb{C}$ don't satisfy the order axioms, but otherwise satisfy the Addition and Multiplication axioms.

You may observe that axiom M5 follows from axioms O1 and O3. The reason it's there is that the number system is typically built up in stages so that a bunch of theorems based on the addition and multiplication axioms can be derived before the order property is introduced, but the condition $0 \neq 1$ is needed for these theorems. Also, (as an aside), one can define number systems that don't have orders (like the complex numbers or the complex 'integers') that still need this axiom.