Test01, Math 5200, September 26, 2025
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NAME: __________

Make sure to show all your work. You may not receive full credit if the accompanying work is incomplete or incorrect. If you do scratch work make sure to indicate scratch work - I will not take off points for errors in the scratch work if it is so labeled.

Note that all the proofs must follow logically from the theorems and definitions stated in the class notes; if you wish to use some lemma that has not been proven in class, you must prove it first using the theorems and definitions stated from the class notes. If you are asked to "prove from the definition" then you must prove the statement from the definition without using any of the theorems.

Problems [80 points]:

Problem 1. Prove, from the definition, that the sequence $\{x_n\}_{n=1}^{\infty}$ has a sequential limit point where:

$$x_n = \frac{5\sqrt{n} - 3}{2\sqrt{n} + 5}.$$

Scratch Work. First we observe that

$$\frac{5m-3}{2m+5} = \frac{5}{2} - \frac{31/2}{2m+5}.$$

Then

$$\begin{aligned}
\frac{31/2}{2m+5} &< \epsilon \\
\frac{1}{2m+5} &< \frac{2\epsilon}{31} \\
2m+5 &> \frac{31}{2\epsilon} \\
2m &> \frac{31}{2\epsilon} - 5 \\
2m &> \frac{31}{2\epsilon} \\
m &> \frac{31}{4\epsilon} \\
\sqrt{n} &> \frac{31}{4\epsilon} \\
n &> \left(\frac{31}{4\epsilon}\right)^2
\end{aligned}$$

□

Proof. We claim that the sequential limit is $\frac{5}{2}$. So, let $\epsilon > 0$. Then we want to find an integer N_ϵ so that if $n > N_\epsilon$ then $|\frac{5}{2} - \frac{5}{2} + \frac{\sqrt{n}-3}{2\sqrt{n}+5}| < \epsilon$. We pick N_ϵ

to be an integer bigger than$(\frac{31}{4\epsilon})^2$, then if $n > N_\epsilon$ we have:

$$\begin{aligned}
n &> \left(\frac{31}{4\epsilon}\right)^2 \\
\sqrt{n} &> \frac{31}{4\epsilon} \\
2\sqrt{n} &> \frac{31}{2\epsilon} \\
2\sqrt{n}+5 &> \frac{31}{2\epsilon} \\
\frac{1}{2\sqrt{n}+5} &< \frac{2\epsilon}{31} \\
\frac{31/2}{2\sqrt{n}+5} &< \epsilon \\
\left|\frac{31/2}{2\sqrt{n}+5}\right| &< \epsilon \\
\left|\frac{5}{2}-\frac{5}{2}+\frac{31/2}{2\sqrt{n}+5}\right| &< \epsilon \\
\left|\frac{5}{2}-x_n\right| &< \epsilon.
\end{aligned}$$

□

Problem 2. Suppose that $\{x_n\}_{n=1}^{\infty}$ is a bounded increasing sequence and $M = \{x_n | n \in \mathbb{Z}^+\}$. Prove that the least upper bound of M is the sequential limit point of the sequence.

Proof. Let L denote the least upper bound of the set M. First we claim that there is no n so that $L = x_n$; for if there were we'd have $L = x_n < x_{n+1}$. This contradicts the fact that L is an upper bound of every element of the sequence. Let $\epsilon > 0$, then since L is the least upper bound and L is not equal to any point in the sequence, there is a point, say x_N, of M greater than $L - \epsilon$. So we have:

$$L - \epsilon < x_N < x_{N+1} < x_{N+2} + \ldots + x_n + \ldots < L.$$

So if $n > N$ we have

$$\begin{aligned}
L - \epsilon &< x_n < L \\
|L - x_n| &< \epsilon.
\end{aligned}$$

□

Problem 3. Prove, from the definition, that if A is the sequential limit point of the sequence $\{a_n\}_{n=1}^{\infty}$ and B is the sequential limit point of the sequence $\{b_n\}_{n=1}^{\infty}$, then $A+B$ is the sequential limit of the sequence $\{a_n+b_n\}_{n=1}^{\infty}$.

Proof. Let $\epsilon > 0$. Then, since A is the sequential limit point of the sequence $\{a_n\}_{n=1}^{\infty}$, there exists an integer N_A so that if $n > N_A$ we have

$$|A - a_n| \;<\; \frac{\epsilon}{2}.$$

Also, since B is the sequential limit point of the sequence $\{b_n\}_{n=1}^{\infty}$, there exists an integer N_B so that if $n > N_B$ we have

$$|B - b_n| \;<\; \frac{\epsilon}{2}.$$

So, if $n > \max\{N_A, N_B\}$ we have

$$|A - a_n| + |B - b_n| \;<\; \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

$$|A + B - a_n - b_n| \leq |A - a_n| + |B - b_n| < \epsilon.$$

□

Problem 4a. Prove that if the least upper bound of the set M is not a member of M then it is a limit point of M.

Proof. Let $\epsilon > 0$. Then, since L is the least upper bound and $L - \epsilon < L$ then $L - \epsilon$ is not an upper bound, some member x of M must be greater than $L - \epsilon$. Since $L \notin M$ and L is an upper bound, we have:

$$L - \epsilon < x < L$$

So $x \neq L$ and $x \in (L - \epsilon, L + \epsilon)$ so L is a limit point of M. □

Problem 4b. Give an example of a bounded set M so the least upper bound of M is not a limit point of M.

Solution. If M is any finite set, then M satisfies the geven condition. □

Bonus take-home problem [Problem 10 of the practice exercises to the test; look up the definition of bounded away from 0 on the practice exercises sheet]:

Prove that if $\{x_n\}_{n=1}^{\infty}$ is a convergent sequence and $M = \{x_n | n \in \mathbb{N}\}$ is bounded away from 0 then the sequence $\{r_n\}_{n=1}^{\infty}$ converges where:

$$r_n = \frac{1}{x_n}.$$

Scratch Work.

$$\begin{aligned}
\left|\frac{1}{x_n} - \frac{1}{p}\right| &= \left|\frac{p - x_n}{px_n}\right| < \epsilon \\
|p - x_n| &< \epsilon |x_n| p \\
|p - x_n| &< \epsilon \delta p.
\end{aligned}$$

□

Proof. Suppose that the sequence $\{x_n\}_{n=1}^{\infty}$ converges to the point p and that the sequence is bounded away from 0. Notice thatt this implies that $p \neq 0$. So there is a constant δ so that $|x_n| > \delta$ for all integers $n > 0$. Suppose that $\epsilon > 0$. Then $\epsilon |p| \delta > 0$ and there exists an integer N so that if $n > N$ then

$$\begin{aligned}
|p - x_n| &< \epsilon |p| \delta \\
\left|\frac{p - x_n}{px_n}\right| &< \frac{\epsilon |p| \delta}{|p||x_n|} \\
\left|\frac{1}{x_n} - \frac{1}{p}\right| &< \epsilon \frac{\delta}{|x_n|} \\
&< \epsilon \cdot 1 = \epsilon.
\end{aligned}$$

Therefore the sequence $\{\frac{1}{x_n}\}_{n=1}^{\infty}$ converges to the value $\frac{1}{p}$. □