Test01, Math 5200, September 26, 2025
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NAME: __________

Make sure to show all your work. You may not receive full credit if the accompanying work is incomplete or incorrect. If you do scratch work make sure to indicate scratch work - I will not take off points for errors in the scratch work if it is so labeled.

Note that all the proofs must follow logically from the theorems and definitions stated in the class notes; if you wish to use some lemma that has not been proven in class, you must prove it first using the theorems and definitions stated from the class notes. If you are asked to "prove from the definition" then you must prove the statement from the definition without using any of the theorems.

Definitions [I'll accept grammatically equivalent statements; 20 points]: *Look up the definitions in the notes.*

1. Define what it means for the function $f : \mathbb{R} \to \mathbb{R}$ to be continuous at the point $(p, f(p))$.

2. Define what it means for the function $f : \mathbb{R} \to \mathbb{R}$ to be uniformly continuous at over the set S.

3. Define what it means for the set S to be compact.

4. Suppose that $\{x_n\}_{n=1}^{\infty}$ is a sequence of real numbeers. What is the meaning of the following notation:

$$\sum_{i=1}^{\infty} x_n.$$

Problems [80 points]:

Problem 1. Prove, from the $\epsilon - \delta$ definition, that the function $f(x) = x^4$ is uniformly continuous over the interval $[0, 2]$.

Solution. Scratch work:

$$\begin{aligned} x^4 - y^4 &= (x^2 - y^2)(x^2 + y^2) \\ &= (x - y)(x + y)(x^2 + y^2). \end{aligned}$$

Observe that the expression $(x + y)(x^2 + y^2)$ is bounded over the interval $[0, 2]$ when $x = y = 2$. So, $(x + y)(x^2 + y^2) \leq 32$.
Proof: Suppose $\epsilon > 0$; then select $\delta = \frac{\epsilon}{32}$. So, if $|x - y| < \delta$ and $x, y \in [0, 2]$ we have:

$$\begin{aligned} |x^4 - y^4| &= |x - y||(x + y)(x^2 + y^2)| \\ &\leq |x - y|32 \\ &< \delta \cdot 32 \\ &< \frac{\epsilon}{32}32 \\ &< \epsilon. \end{aligned}$$

Which is what we needed. $\square$

Problem 2. Show, from the $\epsilon - \delta$ definition that if f and g are continuous functions from the reals to the reals, then so is $7f + \frac{5}{2}g$.

Solution. Let $h(x) = 7f(x) + \frac{5}{2}g(x)$. We will show continuity at the point $(p, h(p)$. Suppose that $\epsilon > 0$. Then there exist numbers δ_1 and δ_2 so that:

$$\begin{aligned} |f(x) - f(p)| < \frac{\epsilon}{14} &\quad \text{for all} \quad |x - p| < \delta_1 \\ |g(x) - g(p)| < \frac{\epsilon}{5} &\quad \text{for all} \quad |x - p| < \delta_2, \end{aligned}$$

If $\delta = \min\{\delta_1, \delta_2\}$, then for $|x-p| < \delta$ we have:

$$\begin{aligned}
|h(x) - h(p)| &= |7f(x) + \frac{5}{2}g(x) - 7f(p) - \frac{5}{2}g(p)| \\
&\leq |7f(x) - 7f(p)| + \left|\frac{5}{2}g(x) - \frac{5}{2}g(p)\right| \\
&\leq 7|f(x) - f(p)| + \frac{5}{2}|g(x) - g(p)| \\
&< 7\frac{\epsilon}{14} + \frac{5}{2} \cdot \frac{\epsilon}{5} \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Which is what we needed.

$\square$

Problem 3. Consider the following sequence of functions:

$$f_n(x) = x^2 + \frac{7x^2}{3\sqrt{n}} \quad \text{for} \quad n \in \mathbb{Z}^+$$

Show that the sequence converges uniformly to a function over the interval $[0, 3]$ and determine the function to which it converges.

Solution. The sequence of functions converges uniformly to $f(x) = x^2$ over the interval $[0, 3]$. Suppose $\epsilon > 0$. Then we need to find a number N so that if $n > N$ then $|f(x) - f_n(x)| < \epsilon$ for all $x \in [0, 3]$. We caim [from scratch work done elsewhere] that $N = \frac{(21)^2}{\epsilon^2}$ works. Let $n > N$, then:

$$\begin{aligned}
\frac{(21)^2}{\epsilon^2} &< n \\
\frac{(21)}{\epsilon} &< \sqrt{n} \\
\frac{(21)}{\sqrt{n}} &< \epsilon \\
\frac{(7 \cdot 9)}{3\sqrt{n}} &< \epsilon \\
\frac{7x^2}{3\sqrt{n}} &< \epsilon \text{ for all } x \in [0,3] \\
\left| x^2 - x^2 - \frac{7x^2}{3\sqrt{n}} \right| &< \epsilon \text{ for all } x \in [0,3] \\
|f(x) - f_n((x)| &< \epsilon \text{ for all } x \in [0,3].
\end{aligned}$$

Which is what we needed.

$\square$

Problem 4a. Prove that that a closed subset of a compct set is compact.

Problem 4b. Argue that the set $M = \{1 + \frac{1}{n} | n \text{ is a positive integer } \}$ is not compact.