Homework Exercise Math 5210.
Due Monday November 18, 2019.

Define the function $f : \mathbb{R} \to \mathbb{R}$ as follows:

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Prove the following two claims.

(A.) The function f is differentiable at $x = 0$.

Proof. The derivative at 0 turns out to be zero (recall my picture on the blackboard). So, with the plan of using the limit definition: let $\epsilon > 0$ and let $\delta > 0$ be such that $\delta < \epsilon$. Then (for $p = 0$) if $|h| < \delta$ we have:

$$\begin{aligned} \left| \frac{f(p+h) - f(p)}{h} - f'(p) \right| &= \left| \frac{h^2 \sin\left(\frac{1}{h}\right) - 0}{h} - 0 \right| \\ &= \left| h \sin\left(\frac{1}{h}\right) \right| \\ &\leq |h| < \delta < \epsilon. \end{aligned}$$

□

(B.) The derivative of f is not continuous.

Proof. For $x \neq 0$ we use the standard differentiation rules (product, chain, quotient) to obtain:

$$\begin{aligned} f'(x) &= 2x \sin\left(\frac{1}{x}\right) + x^2 \cos\left(\frac{1}{x}\right)(-x^{-2}) \\ &= 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right). \end{aligned}$$

We have the following limit calculation (which can be proven similarly to the one calculated above):

$$\lim_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0.$$

Now we observe that:

$$\lim_{n\to\infty} \frac{1}{n\pi} \quad = \quad 0.$$

And if the function f' is continuous at 0 we would have $\lim_{n\to\infty} f'\left(\frac{1}{n\pi}\right) = f'(0)$. But

$$\lim_{n\to\infty, n\in\text{even}} \cos\frac{1}{\frac{1}{n\pi}} \quad = \quad 1.$$

And

$$\lim_{n\to\infty, n\in\text{odd}} \cos\frac{1}{\frac{1}{n\pi}} \quad = \quad -1.$$

So $\lim_{x\to 0} f'(x)$ does not exist.

$\square$