Hand-in Homework Exercise Math 5210.
Due Friday November 22, 2019.

Define the function $f : \mathbb{R} \to \mathbb{R}$ as follows:

$$f(x) = \begin{cases} \frac{1}{3} & \text{if } 0 \leq x < \frac{1}{2} \\ x & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases}$$

The exercise is in two parts:
Part 1: Prove that the function is integrable over $[0, 1]$.
Part 2: Use an ϵ - δ definition of a limit that calculates the integral.

For part 1 I gave the following lemma in class that you may assume and use to prove that the integral exists:

Lemma 1. The function f is integrable over the interval $[a, b]$ if and only if for each $\epsilon > 0$ there exists a δ so that if $P = \{p_0, p_1, p_2, \ldots, p_n\}$ and $Q = \{q_0, q_1, q_2, \ldots, q_m\}$ are partitions of $[a, b]$ so that $\text{mesh}(P) < \delta$ and $\text{mesh}(Q) < \delta$ then for any selections c and d (where $c_i \in [p_{i-1}, p_i]$ and $d_i \in [q_{i-1}, q_i]$) we have:

$$\Big| \sum_{i=1}^{n} f(c_i)(p_i - p_{i-1}) - \sum_{j=1}^{m} f(d_j)(q_j - q_{j-1}) \Big| < \epsilon.$$

I realized after a discussion with a couple of students after class that the following lemma (which follows from exercises 10.2.3 - 10.2.5) might be a more familiar one to use based on Dr. Kilgore's notes. See formulas (10.2) and (10.3) for the definitions of $L(P)$ and $U(P)$ for a partition P.

Lemma 2. The function f is integrable over the interval $[a, b]$ if and only if for each $\epsilon > 0$ there exists a δ so that if $P = \{p_0, p_1, p_2, \ldots, p_n\}$ is a partition so that $\text{mesh}(P) < \delta$ then we have:

$$\Big| U(P) - L(P) \Big| < \epsilon.$$

To do part 1 you may use either of these lemmas without proving them.

For part 2 The following formula will be helpful (or needed):

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$$