Key: Test Math 5210.
November 8, 2019.

Make sure to show all your work. You may not receive full credit if the accompanying work is incomplete or incorrect. Indicate any scratch work that you do - I will not take off points for errors in the scratch work if it is so labeled and will assume that the scratch work is not part of the final answer/proof. Do problem 1 plus four of the remaining five problems. If you do all six I will grade the best five out of six.

In the following $C[a, b]$ denotes the linear space of continuous functions $f : [a, b] \to \mathbb{R}$.

Problem 1. State and prove the Cauchy-Schwartz inequality. Anything equivalent to the following:

$$< u, v >^2 \leq < u, u >< v, v > .$$

Proof. Let u and v be two vectors. From the definition of inner product for vectors t and w we have.

$$\begin{aligned} < t-w, t-w > &\geq 0 \\ < t, t > -2 < t, w > + < w, w > &\geq 0 \\ < t, t > + < w, w > &\geq 2 < t, w > . \end{aligned} \tag{1}$$

Since the above is true for all vectors they will be true for:

$$t = \frac{u}{|u|} = \frac{u}{\sqrt{< u, u >}} \text{ and } w = \frac{v}{|v|} = \frac{v}{\sqrt{< v, v >}}.$$

For these “unit” vectors” we have

$$< t, t >= \left\langle \frac{u}{\sqrt{< u, u >}}, \frac{u}{\sqrt{< u, u >}} \right\rangle = \frac{1}{\sqrt{< u, u >}} \frac{1}{\sqrt{< u, u >}} < u, u >= 1.$$

So equation (1) becomes

$$1 + 1 \geq 2 \left\langle \frac{u}{\sqrt{< u, u >}}, \frac{v}{\sqrt{< v, v >}} \right\rangle$$

canceling the 2's and cross multiplying yields

$$\sqrt{< u,u >< v,v >} \;\; \geq \;\; < u,v > .$$

If $< u,v > ge0$ then square both sides to get the inequality. If $< u,v >< 0$ replace u with $-u$ and again square both side. In either case we have the required inequality. □

Problem 2. Consider the linear space $C[a,b]$

a. State the definition of an inner product on the linear space V.

Solution. This is in section 11.2 in the notes. □

b. Show that the following is an inner product on $C[a,b]$:

$$< f,g >= \int_a^b f(t)g(t)dt.$$

Solution. The only tricky part is showing that $< f,f >= 0$ if and only if $f = 0$ (f is the zero function. Clearly

$$< f,f >= \int_a^b f^2(t)dt \geq 0.$$

Suppose $f \neq 0$ then there is a number $p \in [a,b]$ so that $f(p) \neq 0$ and hence $f^2(p) > 0$. By continuity there is a positive number d so that $0 < d < f^2(p)$ and by continuity there is a $\delta > 0$ so that $f^2(x) > d$ for all $x \in [p-\delta, p+\delta]$. Then by the properties of integration, and since f^2 is non-negative:

$$\int_a^b f^2(t)dt \geq \int_{p-\delta}^{p+\delta} f^2(t)dt \geq \int_{p-\delta}^{p+\delta} d \;\; dt = d(2\delta) > 0.$$

If p is one of the end points, some minor modification of the argument is needed - but I'm not going to worry about that here. □

c. Show that the following is a norm on $C[a,b]$:

$$\|f\| = \sqrt{\int_a^b f^2(t)dt}.$$

Solution. We know that for an inner product $< u, v >$ we can define a norm by

$$|v| = \sqrt{< v, v >}.$$

Either here on in part b you need to have the argument about continuity implying that $\int_a^b f^2(t)dt > 0$ whenever $f^2 \neq 0$. □

Problem 3. Show that the following is a norm on Euclidean 3-space:

$$\|(x, y, z)\| = \sqrt{x^2 + y^2 + z^2}.$$

Solution. It's sufficient to argue that $< (x_1, y_1, z_1), (x_2, y_2, z_2) >= x_1x_2 + y_1y_2 + z_1z_2$ is an inner product. Otherwise some algebraic argument that justifies that the triangular inequality holds is needed. □

Problem 4. Find the a_2 coefficient of the Fourier series of the following function over the interval $[-\pi, \pi]$:

$$f(x) = 5x^2 + 7x.$$

Solution. The a_2 term is the coefficient of the $\cos(2x)$ term. Since $y = 7x$ is an odd function the cosine coefficients are zero. So we have

$$\begin{aligned} a_2 &= \frac{1}{\pi}\int_{-\pi}^{\pi} 5x^2 \cos(2x)dx \\ &= \frac{5}{\pi}\left(x^2\frac{\sin(2x)}{2} + 2x\frac{\cos(2x)}{4} - 2\frac{\sin(2x)}{8}\Big|_{-\pi}^{\pi}\right) \\ &= \frac{5}{\pi}\pi = 5. \end{aligned}$$

□

Problem 5. Suppose that f is a periodic function on the interval $[-L, L]$. Just set up the integral that gives the b_n coefficient of the Fourier series of f over $[-L, L]$.

Solution.

$$b_n = \frac{1}{a}\int_{-a}^{a} f(t)\sin\left(\frac{\pi 2t}{a}\right)dt.$$

□

Problem 6. Argue that if f is one of the functions in the set $\{\sin(nx), \cos(nx)\}_{n=0}^{\infty}$ then the Fourier series of f is the function f itself.

Solution. We have the following identities (make sure you know how to derive them):

$$\begin{aligned}
\int_{-\pi}^{\pi} \sin nx \sin(mx) dx &= 0 \quad \text{for } n \neq m \\
\int_{-\pi}^{\pi} \cos nx \cos(mx) dx &= 0 \quad \text{for } n \neq m \\
\int_{-\pi}^{\pi} \cos nx \sin(mx) dx &= 0 \quad \text{for all } n, m.
\end{aligned}$$

So for any of these functions, the only nonzero a_n or b_n will be for the coefficient of the term that matches the function. E.g. for $\sin(nx)$ we have

$$\int_{-\pi}^{\pi} \sin nx \sin(nx) dx = \pi.$$

The π's cancel when you solve for (in this case) a_n. The calculation is similar for $\cos(nx)$.

You also need to consider $n = 0$. These give you $\sin(0x) = 0$ and $\cos(0x) = 1$, the first is trivial and the second is a quick integration of the constant 1 with π's again canceling. □