Analysis II; Background Notes and Definitions. In regard to the definitions, the a, b or a, b, c of the definitions are meant to be equivalent definitions for the same concept. In each case it would be a theorem that each pair is equivalent. It is likely that different students may have been given different but equivalent definitions.

Notation.

$\mathbb{R}$ denotes the real numbers.

Assume for this sections that the space under consideration is always the reals.

$\mathbb{N}$ denotes the positive integers.

(a, b) denotes the set $\{x|a < x < b\}$; this set is called a segment or an open interval.

$[a, b]$ denotes the set $\{x|a \leq x \leq b\}$; this set is called an interval or (sometimes for emphasis) a closed interval.

Definition 1. The subset U of $\mathbb{R}$ is said to be *open* if and only if $U = \emptyset$ or:

a. for each point $p \in U$ there exists a segment (a, b) so that $p \in (a, b) \subset U$.

b. for each point $p \in U$ there exists a number δ so that $\{x|\ |x - p| < \delta\} \subset U$.

Basic Theorem about open sets: The common part of two open sets is open and the union of an arbitrary collection of open sets is open.

Definition 2. The point p is a *limit point* of the set M. means that:

a. If $S = (a, b)$ is a segment containing p then S contains a point of M distinct from p.

b. If U is an open set containing p then U contains a point of M distinct from p.

c. If $\epsilon > 0$ then there exists a point $x \in M$ so that $0 < |p - x| < \epsilon$.

Definition 3. Let $f : D \to \mathbb{R}$ be a function. Then

$$\lim_{x \to a} f(x) = L$$

means that a is a limit point of D and

a. if U is an open set containing L then there is an open set V containing a so that $f((V - \{a\}) \cap D) \subset U$.

b. if $\epsilon > 0$ there exists a number δ so that if $x \in D$ and $0 < |x - a| < \delta$ then $|f(x) - L| < \epsilon$.

Definition 4. Let $f : D \to \mathbb{R}$ be a function. Then

$$\lim_{x \to \infty} f(x) = L$$

means that D does not have an upper bound and

a. if U is an open set containing L then there is a number B so that $f(\{x|x > B\} \cap D) \subset U$.

b. if $\epsilon > 0$ there exists a number B so that if $x \in D$ and $x > B$ then $|f(x) - L| < \epsilon$.

Observation. Suppose that $s_1, s_2, s_3, \ldots = \{s_n\}_{n=1}^{\infty}$ is a sequence; then the sequential limit of the sequence is defined by considering the function $s : \mathbb{N} \to \mathbb{R}$ defined by $s(n) = s_n$ and using the above definition of limit. Equivalently, the sequence $\{s_n\}_{n=1}^{\infty}$ has sequential limit L means that

a. if U is an open set containing L then there is a number $N \in \mathbb{N}$ so that if $n > N$ then $s_n \in U$.

b. if $\epsilon > 0$ then there is a number $N \in \mathbb{N}$ so that if $n > N$ then $|s_n - L| < \epsilon$.

Definition 5. Suppose $\{s_n\}_{n=1}^{\infty}$ is a sequence of numbers then the series $\sum_{n=1}^{\infty} s_n$ is said to *converge* to the number L means that the sequence $\{\sum_{n=1}^{k} s_n\}_{k=1}^{\infty}$ has sequential limit L.

Definition 6. Suppose that f is a function with domain $D \subset \mathbb{R}$ and range a subset of $\mathbb{R}$. Then f is said to be *continuous at the point* p means that:

a. if U is an open set containing $f(p)$ then there is an open set V containing p so that if $x \in D \cap V$ then $f(x) \in U$.

b. if $\epsilon > 0$ then there exists a number δ so that if $x \in D$ and $|x - p| < \delta$ then $|f(x) - f(p)| < \epsilon$.

c. $\lim_{x \to p} f(x) = f(p)$.

Definition 7. A function is said to be *continuous* if it is continuous at each point of its domain.

Warning! There is at least one statement below denoted as a theorem which is not a theorem. (I.e. the statement is false.) Find it and construct a counter example.

Theorem 0.0 A. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a function. Then f is continuous if and only if for each open set U, $f^{-1}(U)$ is open.

Theorem 0.0 B. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a function. Then f is continuous if and only if for each segment (a, b), $f^{-1}((a, b))$ is open.

Theorem 0.1 A. Suppose that $\{a_n\}_{n=1}^{\infty}$ is a sequence with sequential limit a and $\{b_n\}_{n=1}^{\infty}$ is a sequence with sequential limit b. Then the sequence $\{a_n + b_n\}_{n=1}^{\infty}$ has sequential limit $a + b$.

Theorem 0.1 B. Suppose that the series $\sum_{n=1}^{\infty} a_n$ converges to a and the series $\sum_{n=1}^{\infty} b_n$ converges to b. Then the series $\sum_{n=1}^{\infty}(a_n + b_n)$ converges to $a + b$.

Theorem 0.1 C. Suppose that each of f and g is a function with common domain D and that both f and g are continuous at the point p. Then the function $f + g$ is continuous at the point p.

Theorem 0.2 A. Suppose that $\{a_n\}_{n=1}^{\infty}$ is a sequence with sequential limit a and $\{b_n\}_{n=1}^{\infty}$ is a sequence with sequential limit b. Then the sequence $\{a_n b_n\}_{n=1}^{\infty}$ has sequential limit ab.

Theorem 0.2 B. Suppose that the series $\sum_{n=1}^{\infty} a_n$ converges to a and the series $\sum_{n=1}^{\infty} b_n$ converges to b. Then the series $\sum_{n=1}^{\infty} a_n b_n$ converges to ab.

Theorem 0.2 C. Suppose that each of f and g is a function with common domain D and that both f and g are continuous at the point p. Then the function $f \cdot g$ is continuous at the point p.

Theorem 0.3 A. Suppose that $\{a_n\}_{n=1}^{\infty}$ is a sequence with sequential limit a, and $a \neq 0$. Then there is an integer N so that the sequence $\{\frac{1}{a_n}\}_{n=N}^{\infty}$ has sequential limit $\frac{1}{a}$.

Theorem 0.3 B. Suppose that the series $\sum_{n=1}^{\infty} a_n$ converges to a, $a \neq 0$, and $a_n \neq 0$ for all $n \in \mathbb{N}$. Then the series $\sum_{n=1}^{\infty} \frac{1}{a_n}$ converges to $\frac{1}{a}$.

Theorem 0.3 C. Suppose that f is a function with domain D, that $f(x) \neq 0$ for all $x \in D$ and that f is continuous at the point p. Then the function $\frac{1}{f}$ is continuous at the point p.

Definition 7. A set is said to be *closed* if and only if its complement is open.

Definition 8. The collection of sets G is said to *cover* the set M if and only if every point in M lies in some element of G.

Definition 9. The set M is said to be *compact* if and only if whenever G is a collection of open sets covering M then there is a finite subcollection of G that covers M.

Theorem 0.4. The set M is closed if and only if it contains all of its limit points.

Theorem 0.5 (Bolzano-Weierstrass). If M is a bounded infinite set then there is a point that is a limit point of M.

Theorem 0.6 (Heine-Borel). The set M is compact if and only if it is closed and bounded.

Theorem 0.7. If $f : \mathbb{R} \to \mathbb{R}$ is continuous and M is compact, then $f(M)$ is compact.

Theorem 0.8 (Intermediate value theorem). Suppose that $f : [a, b] \to \mathbb{R}$ is continuous and c is between $f(a)$ and $f(b)$. Then there is a point z so that $a < z < b$ and $f(z) = c$.

Comment: Theorems 0.5, 0.6 and 0.8 require the least upper bound axiom or some equivalent statement.

Theorem 0.9 (High point theorem). If M is compact and $f : M \to \mathbb{R}$ is continuous, then there is a number $c \in M$ so that if $x \in M$ then $f(x) \leq f(c)$.