The Derivative.

Reminder: If $D \subset R$ then $Int(D)$ denotes the set to which x belongs if and only if there is an open set U so that $x \in U \subset D$.

Definition. Suppose that $f : D \to \mathbb{R}$ is a function and $p \in Int(D)$. Then f is said to be *differentiable* at p means that there exists a number m so that

$$\lim_{h \to 0} \frac{f(p+h) - f(p)}{h} = m.$$

The number m is called the derivative of f at the point p and m is denoted by $f'(p)$.

Theorem 1.1. If the function f is differentiable at the point p then it is continuous at the point p.

Question. Is the converse of Theorem 1.1 true?

Theorem 1.2 A. The function $f : D \to \mathbb{R}$ is differentiable at the point p if and only if:

$$\lim_{x \to p} \frac{f(p) - f(x)}{p - x} = f'(p).$$

Theorem 1.2 B [The geometric picture of the derivative]. Suppose that $f : D \to \mathbb{R}$ is a function and $p \in Int(D)$. Then f is differentiable at p if and only it there exists a number m so that if U is an open set containing $f(p)$ and α is a line containing $(p, f(p))$ with slope less than m and β is a line containing $(p, f(p))$ with slope bigger than m then there is an open set V containing p so that if $x \in V - \{p\}$ then $f(x)$ lies between α and β. Furthermore $m = f'(p)$.

Theorems 1.3. Suppose that f and g are functions whose domains contain p in their interiors, c is the constant function and all the quantities in the following formulas are defined. Then:

$$c'(p) = 0 \qquad (1)$$

$$
\begin{aligned}
(cf)'(p) &= cf'(p) && (2)\\
(f+g)'(p) &= f'(p)+g'(p) && (3)\\
(fg)'(p) &= f'(p)g(p)+f(p)g'(p) && (4)\\
\left(\frac{1}{f}\right)'(p) &= -\frac{f'(p)}{(f(p))^2} && (5)\\
(f\circ g)'(p) &= f'(g(p))g'(p) && (6)
\end{aligned}
$$

Theorem 1.4. Suppose that $r \in R$ and f is defined by $f(x) = x^r$. Then $f'(x) = rx^{r-1}$.

[Hint: Prove versions of the theorem for the following cases:

(i.) $r \in \mathbb{N}$;
(ii.) $r \in \mathbb{Z}$;
(iii.) $\frac{1}{r} \in \mathbb{Z}, r \neq 0$;
(iv.) $r \in \mathbb{Q}$;
(v.) $r \notin \mathbb{Q}$.]

Theorem 1.5 (Rolle's theorem, Relative Max/Min theorem). Suppose that f is a function that is differentiable at p and that there is an open set V containing p so that $f(x) \leq f(p)$ for all $x \in V$. Then $f'(p) = 0$.

Lemma to Theorem 1.5. If $f : [a,b] \to \mathbb{R}$ is a function, $a < p < b$ and $f'(p) \neq 0$, then if $\delta > 0$ there is a point $q \in (p-\delta, p+\delta)$ so that $f(q) > f(p)$.

Theorem 1.5B (Rolle's theorem, Relative Max/Min theorem). Suppose that f is a function that is differentiable at p and that there is an open set V containing p so that $f(x) \geq f(p)$ for all $x \in V$. Then $f'(p) = 0$.

Corollary to Theorem 1.5. Suppose that $f : [a,b] \to \mathbb{R}$ is a continuous function that is differentiable at each point of (a,b). Then there is a point p between a and b so that $f'(p) = 0$.

Question. If f is differentiable at each point of a segment (a,b) then must f be continuous on (a,b).

Theorem 1.6 (Mean value theorem). Suppose that f is continuous on the interval $[a,b]$ and is differentiable at each point of the segment (a,b). Then there is a point c between a and b so that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Theorem 1.7 (Intermediate value theorem for derivatives). Suppose that f is differentiable on the interval $[a, b]$ and m is between $f'(a)$ and $f'(b)$. Then there is a point z between a and b so that $f'(z) = m$.