The R-integral

Definition. Let $[a, b]$ be an interval. Then S is a subdivision of $[a, b]$ means that S is a finite increasing sequence $\{x_i\}_{i=0}^n$ so that $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$.

Definition. If $S = \{x_i\}_{i=0}^n$ is a subdivision of the interval $[a, b]$ then $\text{mesh}(S) = \sup\{(x_i - x_{i-1}) \mid 0 < i \leq n\}$.

Definition. Suppose that $[a, b]$ is an interval and $f : [a, b] \to \mathbb{R}$ is a function. Then f is integrable over the interval $[a, b]$ means that there is a number I so that if $\epsilon > 0$ is a positive number, then there exists a positive number $\delta > 0$ so if $S = \{x_i\}_{i=1}^n$ is a subdivision of $[a, b]$ with $\text{mesh}(S) < \delta$ and for each i, $x_i^* \in [x_{i-1}, x_i]$ then

$$\Big|\sum_{i=1}^{n} f(x_i^*)(x_i - x_{i-1}) - I\Big| < \epsilon.$$

The number I is called the integral of the function f over the interval $[a, b]$ and it is denoted by

$$\int_a^b f(x)dx \text{ or } \int_{[a,b]} f.$$

We will use both notations.

Exercise 2.0. (a.) Show that the number I in the definition of the integral is unique.

(b.) Show that if f is integrable on $[a, c]$ and $a < b < c$ then f is integrable on $[a, b]$.

Theorem 2.1. If $f : [a, b] \to \mathbb{R}$ is a continuous function, then it is integrable over the interval $[a, b]$.

Exercise 2.1. In the following exercises the function f is defined for a particular interval $[a, b]$. Determine from the definition if $\int_{[a,b]} f$ exists and if it does, determine its value.

$$\begin{array}{lll} a.) & f(x) = 1, & x \in [a, b]. \\ b.) & f(x) = x, & x \in [0, 1]. \\ c.) & f(x) = x^2, & x \in [0, 1]. \end{array}$$

$$d.) \quad f = \begin{cases} 1 & \text{if } 0 \le x < 1 \\ 3 & \text{if } 1 \le x \le 2 \end{cases} \qquad x \in [0, 2].$$

$$e.) \quad f = \begin{cases} 0 & \text{if } x = 0 \\ \frac{1}{x} & \text{if } 0 < x \le 1 \end{cases} \qquad x \in [0, 1].$$

$$f.) \quad f = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases} \qquad x \in [0, 1].$$

Exercise 2.2. If f is integrable over the interval $[a, b]$ then:

$$\int_{[a,b]} f = \lim_{n\to\infty} \sum_{i=1}^{n} f\Big(a + i\frac{b-a}{n}\Big)\frac{b-a}{n}.$$

Is the converse true (i.e. if the limit on the right exists then is f integrable)?

Notation: If $b < a$ then $\int_{[a,b]} f = -\int_{[b,a]} f$; $\int_a^a f(x)dx = 0$.

Theorem 2.2. Suppose that f and g are continuous functions whose domains include the intervals over which they are integrated, c is the constant function and all the quantities in the following formulas are defined. Then:

$$\int_a^b cf(x)dx = c\int_a^b f(x)dx \tag{1}$$

$$\int_a^b f(x) + g(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx \tag{2}$$

$$\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx \tag{3}$$

Theorem 2.3. Suppose that f and g are continuous functions whose domains include the interval $[a, b]$ and that $f(x) \leq g(x)$. Then

a. $\int_{[a,b]} f \leq \int_{[a,b]} g$.

b. If there is a value $c \in [a, b]$ so that $f(c) < g(c)$ then $\int_{[a,b]} f < \int_{[a,b]} g$.

Theorem 2.4. Suppose that f is a continuous function whose domain includes the interval $[a, b]$ and that for each number $a \leq x \leq b$ we define $F(x) = \int_{[a,x]} f$. Then

a. F is continuous for each $x \in [a, b]$;

b. F is differentiable for each $x \in [a, b]$;

c. [The fundamental theorem of calculus.] $F'(t) = f(t)$ for each $t \in [a, b]$.

Theorem 2.5. Suppose that f is a continuous function whose domain includes the interval $[a, b]$ then there exists a number c with $a < c < b$ so that

$$\int_a^b f(x)dx = f(c)(b - a).$$

Exercise 2.3. Suppose that f and g are continuous functions whose domains include the interval $[a, b]$, $a < c < b$ and the function h is defined by:

$$h(x) = \begin{cases} f(x) & \text{if } a \leq x < c \\ g(x) & \text{if } c \leq x \leq b. \end{cases}$$

Then h is integrable over $[a, b]$ and $\int_{[a,b]} h = \int_{[a,c]} f + \int_{[c,b]} g$.