Integrability, Lemmas and Hints

Theorem 2.1b: Suppose that $f : [a, b] \to \mathbb{R}$ is a function. If the function f is R-integrable, then it is bounded.

Hint: If f is not bounded over $[a, b]$ then there exists a number $c \in [a, b]$ so that if B is a number and $\delta > 0$ then there is a point $x \in (c - \delta, c + \delta)$ so that $|f(x)| > B$.

Definition: Let S be a subdivision with $S : \{a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b\}$ and let f be a function. For this particular subdivision S and function f, define for each i,

$$\begin{aligned} y_i^m &= \text{glb}\{f(x_i^*) \mid x_i^* \in [x_{i-1}, x_i]\} \\ y_i^M &= \text{lub}\{f(x_i^*) \mid x_i^* \in [x_{i-1}, x_i]\}. \end{aligned}$$

Then for the function f define

$$\begin{aligned} \text{Lower Sum}(S, f) &= \sum_{i=1}^{n} y_i^m (x_i - x_{i-1}) \\ \text{Upper Sum}(S, f) &= \sum_{i=1}^{n} y_i^M (x_i - x_{i-1}). \end{aligned}$$

Lemma 2.2b. Let S be a subdivision with $S : \{a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b\}$ and let $f : [a, b] \to \mathbb{R}$. Then f is R-integrable over [a,b] if and only if it is true that if $\epsilon > 0$ then there exists a number $\delta > 0$ so that if S is a subdivision of $[a, b]$ with mesh less than δ then

$$\text{Upper Sum}(S, f) - \text{Lower Sum}(S, f) \ < \ \epsilon.$$

Hint: For each $n \in \mathbb{N}$ let S_n be a subdivision of $[a, b]$ with $\text{mesh}(S_n) < \frac{1}{n}$. Argue that the sequential limit of $\{\text{Lower Sum}(S_n, f)\}_{n=1}^{\infty}$ is $\int_{[a,b]} f$. (And observe that $\{\text{Upper Sum}(S_n, f)\}_{n=1}^{\infty}$ also converges to $\int_{[a,b]} f$.)

Lemma 2.3b. If $f : [a, b] \to \mathbb{R}$ is continuous then f satisfies the hypothesis of Lemma 2.2b.

Hint: Use the uniform continuity theorem.

Claim: Lemmas 2.2b and 2.3b imply Theorem 2.1.

Lemma 2.4b. Suppose $f : [a, b] \to \mathbb{R}$ is continuous and $g : [b, c] \to \mathbb{R}$ is continuous and h is defined as follows:

$$h(x) = \begin{cases} f(x) & \text{if } a \leq x < b \\ d & \text{if } x = b \\ g(x) & \text{if } b < x \leq c. \end{cases}$$