Properties of the Integral

In the following make the usual assumptions: a, b, c denote numbers, all functions are assumed to map into the reals, etc. When I use the term "integrable" I mean R-integrable according to our definition.

Theorem 3.0 If f is integrable over the interval $[a, b]$ and $a < c < b$, then

$$\int_{[a,b]} f = \int_{[a,c]} f + \int_{[c,b]} f.$$

[This is an important theorem; you may assume it as needed in the following.]

Theorem 3.1. If f is integrable over the interval $[a, b]$ then:

$$\int_{[a,b]} f = \lim_{n\to\infty} \sum_{i=1}^{n} f\Big(a + i\frac{b-a}{n}\Big)\frac{b-a}{n}.$$

[Hint: do the case where $f(x) \geq 0$ for all $x \in [a, b]$ first.]

Theorem 3.2. If f is integrable over the interval $[a, b]$ then f is bounded.

Lemma to 3.2. Suppose that $f : [a, b] \to \mathbb{R}$ is an unbounded function. Then there exist a point $\ell \in [a, b]$ so that if $p < \ell < q$ then $f|_{[p,q]}$ is unbounded. [I.e. f is unbounded on every open set containing ℓ. Note that I should have had $[p, q] \cap [a, b]$ in the domain of f above.]

[Hint (for thm and lemma): again, do the case where $f(x) \geq 0$ for all $x \in [a, b]$.]

Theorem 3.3 If $f : [a, b] \to \mathbb{R}$ is continuous, then if $\epsilon > 0$ there exists a number $\delta > 0$ so that if S is a subdivision of $[a, b]$ with mesh less than δ then

$$\text{Upper Sum}(S, f) - \text{Lower Sum}(S, f) \;<\; \epsilon.$$

Note: the purpose of this theorem is that once we prove the following theorem we get theorem 2.1. Also, to get a good understanding of the integral, consider proving theorem 3.3' without using theorem 3.1.

Theorem 3.3' If $f : [a,b] \to \mathbb{R}$ is continuous, then f is integrable over $[a,b]$ if and only if for each $\epsilon > 0$ there exists a number $\delta > 0$ so that if S is a subdivision of $[a,b]$ with mesh less than δ then

$$\text{Upper Sum}(S,f) - \text{Lower Sum}(S,f) \quad < \quad \epsilon.$$

[Hint: work through the 'easy" way first.]

Theorem 3.4. Suppose each of f and g is integrable over the interval $[a,b]$ then:

1. for a number c, cf is integrable over the interval $[a,b]$; furthermore
$$\int_{[a,b]} cf = c\int_{[a,b]} f.$$
2. $f+g$ is integrable over the interval $[a,b]$; furthermore
$$\int_{[a,b]} f+g = \int_{[a,b]} f + \int_{[a,b]} g.$$
3. $f \cdot g$ is integrable over the interval $[a,b]$; furthermore if each of f' and g' is continuous then
$$f(b)\cdot g(b) - f(a)g(a) = \int_{[a,b]} f\cdot g' + \int_{[a,b]} f'\cdot g.$$
4. If $f(x) \neq 0, x \in [a,b]$, $\frac{1}{f}$ is integrable over the interval $[a,b]$.
5. If range$(g) \subset$ domain(f) then $f \circ g$ is integrable over an interval in the domain of g.

Comments [re Theorem 3.4]: One of these is false (so it's not a theorem (yet)), add sufficient conditions for it to be true.

Exercises: See if you can do the parts of Theorem 3.4 from the $\epsilon - \delta$ definition without recourse to Theorem 3.1. Following are some hints.

1. This is an easy $\epsilon - \delta$ argument. You may have to consider two cases: $c > 0$ and $c < 0$.

2. This is another standard $\epsilon - \delta$ argument.

3. This is the integration by parts formula - and we all "know" it's true from calculus. But it looks like we need to assume continuity of the derivatives. We should construct an example that makes that necessary. Implicit in the theorem is the fact that if f is integrable and g' is integrable, then fg' is integrable. Note that the integration by parts formula follows easily from the "fundamental theorem."

4. consider the case where f is continuous on $[a, b]$ first.

Definition. The function $f : [a, b] \to \mathbb{R}$ is *non-decreasing* means that if x_1, x_2 are points in the domain of f and $x_1 < x_2$ then $f(x_1) \leq f(x_2)$.

Theorem 3.5. Suppose that $f : [a, b] \to \mathbb{R}$ is non-decreasing. Then f is R-integrable on the interval $[a, b]$.
[Hint: We've already done the "increasing" case.]

Theorem 3.6. Suppose $f : [a, b] \to \mathbb{R}$ is a bounded function; then:

1. If f is discontinuous at exactly one point, then f is integrable.

2. If f is discontinuous at finitely many points, then f is integrable.

3. If the set of discontinuities of f has exactly one limit point, then f is integrable.

4. If the set of discontinuities of f ... (fill in something more complicated.)

Theorem 3.7. Suppose that $f : [a, b] \to \mathbb{R}$ is integrable over $[a, b]$. Then $F : [a, b] \to \mathbb{R}$ defined by

$$F(x) = \int_a^x f(t)dt$$

is continuous.

Where $\int_a^a f(t)dt$ is defined to be 0.

[Note that, in the case that f is continuous, this follows trivially from the Fundamental Theorem which states that $F'(x) = f(x)$ - that theorem is coming.]