Bounded Variation

Reminder Exercise 4.2. Suppose that for each integer n, S_n is a subdivision of $[a,b]$ with mesh$(S_n) < \frac{1}{n}$. Then then if $f : [a,b] \to \mathbb{R}$ is a function and for each integer n, $R^u(f, S_n)$ and $R^\ell(f, S_n)$ are the upper and lower Riemann sums for the function f for the subdivision S_n then f is integrable if and only if

$$\lim_{n\to\infty} R^u(f, S_n) - R^\ell(f, S_n) = 0.$$

Definition. Suppose that $f : [a,b] \to \mathbb{R}$ is a function. Then $\mathcal{V}_a^b(f)$ is the total variation of f over the interval $[a,b]$ means that:

$$\mathcal{V}_a^b(f) = \sup\left\{ \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \,\middle|\, a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b \right\}.$$

If $\mathcal{V}_a^b(f)$ is finite, then f is said to be of bounded variation.

Exercise 4.3. If $f : [a,b] \to \mathbb{R}$ is non-decreasing then f is of bounded variation.

Theorem 4.4. If $f : [a,b] \to \mathbb{R}$ is non-decreasing and $g : [a,b] \to \mathbb{R}$ is non-increasing then $f - g$ is of bounded variation.

Definition. A function $f : [a,b] \to \mathbb{R}$ is said to be monotone if and only if it is either non-increasing or non-decreasing.

Theorem 4.5. The function $f : [a,b] \to \mathbb{R}$ is of bounded variation if and only if it is the difference of two monotone functions.

Definition. Suppose $f : [a,b] \to \mathbb{R}$ is a function which is not continuous at the point p and that both $\lim_{x\to p^-} f(x)$ and $\lim_{x\to p^+} f(x)$ exist and are finite. Then f is said to have a jump discontinuity at p.

Exercise 4.4. Suppose $f : [a,b] \to \mathbb{R}$ is a non-decreasing function which is not continuous at the point p. Then f has a jump discontinuity at p.

Exercise 4.5.

a. If $f : [a,b] \to \mathbb{R}$ is continuous then is it of bounded variation?

b. If $f : [a, b] \to \mathbb{R}$ is integrable then is it of bounded variation?

Exercises 4.6.

a. Show that the following function is not of bounded variation on the interval $[0, 1]$:

$$f = \begin{cases} \frac{1}{x} & \text{if } x > 0 \\ 0 & \text{if } x = 0 \end{cases}$$

b. Show that if the function f is of bounded variation over the interval $[a, b]$ then it is bounded over $[a, b]$.

c. Show that the function $f(x) = (1 - x)x$ is of bounded variation over $[0, 1]$; find the total variation of f over $[0, 1]$.

d. Find a bounded function that is not of bounded variation over some interval.

e. Find a continuous function that is not of bounded variation over some interval.

Theorem 4.6. If $f : [a, b] \to \mathbb{R}$ is of bounded variation then f is integrable.

Theorem 4.7. If $f : [a, b] \to \mathbb{R}$ is Riemann integrable and $g : [c, d] \to [a, b]$ is of bounded variation then f is Riemann-Stieltjes integrable with respect to g.

Theorem 4.8. Suppose that $g : [c, d] \to [a, b]$ is a function so that every Riemann integrable function is Riemann-Stieltjes integrable with respect to g. Then g is of bounded variation over $[c, d]$.

[Caveat: I found versions of theorems 4.5 and 4.8 on the internet and haven't had time to check if they are correct. My intuition tells me they're probably correct, but given the unreliability of the internet (not to mention my intuition) there's a chance they are not correct.]