Bounded Variation and Nowhere Dense Sets.

Definition. The subset M of $\mathbb{R}$ is said to be *nowhere dense* means that if U is a non-empty open set then there is a non-empty open subset V of U that does not intersect M.

[Note: the word "segment" can replace the word "open set" in the above and the definition is equivalent.]

Exercises: Determine which of these sets are nowhere dense:

(a) The integers.
(b) A finite set.
(c) The rational numbers.
(d) $\{\frac{1}{n}|n \in \mathbb{N}\}$.
(e) An open set.
(f) The boundary of an open set.

Theorem 4.9. If M is a nowhere dense then $\overline{M}$ is nowhere dense.

Some lemmas:

(i) The set M is nowhere dense if and only if $\overline{M}^c$ is a dense open set.

(ii) If the set M is not nowhere dense then there exists an open set U so that if $V \subset U$ is open and non-empty, then $V \cap M \neq \emptyset$.

(iii) If the set M is not nowhere dense then there exists an open set U so that every point of U is a limit point of M.

Theorem 4.10. If M is is an interval $[a, b]$, then M is not the union of countable many nowhere dense sets. [Hint: recall that the monotonic common part of non-empty compact sets is not empty.]

Definition. The subset M of the space X is said to be *perfect* if and only if every point of M is a limit point of M.

Theorem 4.11. There exists a closed perfect nowhere dense subset of the reals.

Lemma 4.12. Suppose $f : [a, b] \to \mathbb{R}$ is an increasing function and M is the set of numbers in the domain of f at which f is discontinuous. Then $\{f(x)|x \in M\}$ is nowhere dense in $\mathbb{R}$.

Lemma 4'. Same as lemma 4 except replace "increasing" with "non-decreasing".

Theorem 4.13. Suppose $f : [a, b] \to \mathbb{R}$ is an increasing function. Then f is continuous at some point. Furthermore, if $M = \{x | f \text{ is continuous at } (x, f(x))\}$ then M is dense in $[a, b]$

Theorem 4.14. Suppose $f : [a, b] \to \mathbb{R}$ is a bounded variation function. Then f is continuous at some point. Furthermore, if $M = \{x | f \text{ is continuous at } (x, f(x))\}$ then M is dense in $[a, b]$.