Riemann-Stieltjes Integral

Definition. Let $\{f_n\}_{n=1}$ be a sequence of functions. Then the sequence of functions converges uniformly to the function f on the interval $[a, b]$ means that if $\epsilon > 0$ is a positive number then there exists an integer N so that

$$|f(x) - f_n(x)| < \epsilon \quad \text{for all} \quad n > N, x \in [a, b]$$

Review exercise 4.1. Consider the sequence $f_n(x) = x^n$. Show that:
(a) the sequence does not converge uniformly on $[0, 1]$;
(b) the sequence does converge uniformly on $[0, \frac{1}{2}]$;
(c) $\int_0^1 f_n$ converges.

Background Theorem. Let $\{f_n\}_{n=1}$ be a sequence of continuous functions that converges uniformly to the function f on the interval $[a, b]$. Then f is continuous.

Theorem 4.1. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of continuous functions each of which is integrable over the interval $[a, b]$ that converges uniformly to the function f on the interval $[a, b]$. Then f is integrable over the interval $[a, b]$.

Exercise 4.1a. Under the hypothesis of Theorem 4.1, show that $\{\int_a^b f_n\}_{n=1}^{\infty}$ converges to $\int_a^b f$.

Exercise 4.1b. Suppose that the hypothesis of Theorem 4.1 is changed as follows:

Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions each of which is integrable over the interval $[a, b]$ that converges uniformly to the function f on the interval $[a, b]$. Then is f integrable over the interval $[a, b]$? What if the additional condition that each f_n be increasing is added?

Exercise 4.2. Show that if "integrable" is replaced with "differentiable" in Theorem 4.1, then it is not a theorem. I.e. find a sequence of differentiable functions that converges uniformly to a non-differentiable function.

Definition. Suppose that $[a, b]$ is an interval and $f : \mathbb{R} \to \mathbb{R}$ is a function and $g : [a, b] \to \mathbb{R}$. Then f is integrable over the interval $[a, b]$ with respect to the function g means that there is a number I so that if $\epsilon > 0$ is a positive number, then there exists a positive number $\delta > 0$ so that if $S = \{t_i\}_{i=1}^n$ is a subdivision of $[a, b]$ with $\text{mesh}(S) < \delta$ and for each i, $t_i^* \in [t_{i-1}, t_i]$ then

$$\Big| \sum_{i=1}^{n} f(t_i^*)(g(t_i) - g(t_{i-1})) - I \Big| < \epsilon.$$

Notation: The number I in the definition is denoted by $\int_{[a,b]} f dg$ or by $\int_a^b f dg$.

Exercise 4.3. Let $f(x) = 5$ let $g(x) = x^2$. Calculate $\int_0^1 f dg$ and show from the definition that your calculation is correct.

Hint:

$$x_i^2 - x_{i-1}^2 = (x_i + x_{i-1})(x_i - x_{i-1}).$$

Exercise 4.3b. Let g be defined as follows:

$$g(x) = \begin{cases} 1 & \text{if } x \le 1 \\ 3 & \text{if } 1 < x. \end{cases}$$

Calculate $\int_0^2 f dg$ for the following functions f:

i) $f(x) = x$;

ii.) $f(x) = x^2$;

iii.) f is continuous;

iv.) $f = g$.

v.) repeat i- iv with g defined as follows:

$$g(x) = \begin{cases} x & \text{if } x \le 1 \\ x+1 & \text{if } 1 < x. \end{cases}$$

Exercise 4.3c. Let $g(x) = x^2$. Calculate $\int_0^1 f dg$ for the following functions f and show from the definition that your calculation is correct.

$$\begin{aligned} (i.) &= f(x) = 5x \\ (ii.) &= f(x) = 5x^2. \end{aligned}$$

Theorem 4.2. If $f : \mathbb{R} \to \mathbb{R}$ is a continuous function and $g : [a, b] \to \mathbb{R}$ is a non-decreasing function then f is integrable with respect to g over $[a, b]$.

Furthermore, under suitable conditions:

$$\int_a^b f dg = \int_a^b f g' dx.$$

[Hint/Exercise: See if you can prove Theorem 4.2 under the special condition that f is a constant and g is increasing.]

Theorem 4.3. Let $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ be a sequences of continuous functions that converge uniformly to the functions f and g respectively on the intervals $[g(a), g(b)]$ and $[a, b]$ respectively. Then, under suitable hypothesis, (e.g. all the functions involved have continuous derivatives, though some are true under less restrictive hypothesis, the domains of the f_n's match the ranges of the g_n's):

a. $\{\int_a^b f_n dg_n\}_{n=1}$ converges to $\int_a^b f dg$.

b. Let $F_n(x) = \int_a^x f_n(x)dx$ and $F(x) = \int_a^x f(x)dx$ then $\{F_n\}_{n=1}^{\infty}$ converges uniformly to the functions F on $[a, b]$.

Helpful reminders and observations.

Definition. The sequence $\{x_n\}_{n=1}^{\infty}$ is called a Cauchy sequence if and only if whenever $\epsilon > 0$ there is an integer N so that if $n, m > N$ then

$$|x_n - x_m| < \epsilon.$$

Background theorem. If $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence then it converges.

If the function f is not continuous then there may not be a point in an interval where the function attains it's maximum. However, an application of the lub property should yield the following.

Lemma. Suppose that f is integrable over $[a, b]$, $S = \{t_i\}_{i=1}^{n}$ is a subdivision and B_i denotes the least upper bound of $\{f(x)|x \in [t_{i-1}, t_i]\}$. Then there is a positive number δ so that if S is a subdivision of $[a, b]$ with $\text{mesh}(S) < \delta$ then

$$\Big|\sum_{i=1}^{n} B_i(t_i - t_{i-1}) - \int_a^b f\Big| < \epsilon.$$