Power Series

Definition. Suppose that $\{a_i\}_{i=1}^{\infty}$ is a sequence. Then

$$\sum_{i=1}^{\infty} a_i \quad = \quad L$$

means that the sequence of partial sums

$$\left\{\sum_{i=1}^{n} a_i\right\}_{n=1}^{\infty}$$

has sequential limit L. Such a series is said to converge; a series for which no such limit exists is said to diverge.

Theorem 5.1. Suppose that the series $\sum_{n=0}^{\infty} a_n$ converges. Then

$$\lim_{n\to\infty} |a_n| = 0.$$

Exercise. Show that the converse to Theorem 5.2 is not true.

Theorem 5.2. Let r be a number then, the series

$$\sum_{n=0}^{\infty} r^n$$

converges if and only if $|r| < 1$. Furthermore, if $|r| < 1$ then

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}.$$

Theorem 5.3. If the series $\sum_{n=0}^{\infty} |a_n|$ converges, then so does $\sum_{n=0}^{\infty} a_n$.

Exercise. Show that the converse to Theorem 5.3 is not true.

Definition. $\int_K^{\infty} f dg$ means the following limit if it exists:

$$\lim_{n\to\infty} \int_K^n f dg.$$

Theorem 5.4 [The integral test]. Suppose that the function f is defined for all positive integers and that $f|_{[K,\infty)}$ is defined and is positive and decreasing. Then $\sum_{n=1}^{\infty} f(n)$ converges if and only if $\int_K^{\infty} f$ exists.

Theorem 5.5 [The comparison test]. Suppose that $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are two sequences so that $0 \leq a_n \leq b_n$ for all $n \in \mathbb{N}$. Then:

1.) If $\sum_{n=1}^{\infty} b_n$ converges, then so does $\sum_{n=1}^{\infty} a_n$;
1.) If $\sum_{n=1}^{\infty} a_n$ diverges, then so does $\sum_{n=1}^{\infty} b_n$.

Theorem 5.6 [The limit comparison test]. Suppose that each of $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ is a series of positive numbers and that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L \neq 0.$$

Then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} b_n$ converges.

Exercise. If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ then something still can be said about the relationship between the two series. What is that?

Theorem 5.7 [The ratio test]. Consider the series $\sum_{n=0}^{\infty} a_n$ and let

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|.$$

Then:

$$\begin{array}{ll} \text{If} \quad L < 1 & \text{then the series converges} \\ \text{If} \quad L > 1 & \text{then the series diverges.} \end{array}$$

Exercise. Let $\sum_{n=0}^{\infty} a_n$ be a series so that $a_n > 0$ and suppose

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1.$$

(i.) Find an example where the series $\sum_{n=0}^{\infty} a_n$ converges;
(ii.) Find an example where the series $\sum_{n=0}^{\infty} a_n$ diverges.

Power Series

Theorem 5.8. Suppose that $\{A_n\}_{n=1}^{\infty}$ is a sequence of numbers and

$$\lim_{n\to\infty}\left|\frac{A_n}{A_{n+1}}\right| = r.$$

Then if $|x| < r$ the series $\sum_{n=1}^{\infty} A_n x^n$ converges.

For the following theorems assume that $\{A_n\}_{n=1}^{\infty}$ is a sequence of numbers, $r < 1$ is a number so that

$$\lim_{n\to\infty}\left|\frac{A_n}{A_{n+1}}\right| = r$$

and f is defined by

$$f(x) = \sum_{n=0}^{\infty} A_n x^n \text{ for } -r < x < r.$$

Theorem 5.9. If $0 < \delta < r$, then then sequence of functions $f_n = \sum_{i=0}^{n} A_i x^i$ converges uniformly to the function f on the interval $[-\delta, \delta]$.

Theorem 5.10. If $0 < x < r$, then

$$\int_0^x f(t)dt = \sum_{n=0}^{\infty} A_n \frac{x^{n+1}}{n+1}.$$

Theorem 5.11. If $0 < x < r$, then

$$f'(x) = \sum_{n=1}^{\infty} A_n n x^{n-1}.$$