Measure and the Lebesgue Integral.

In order to consider the Lebesgue integral we need to define the concept of measure and prove some of its properties. Unless otherwise stated assume that all sets are subsets of some given segment, say (α, β).

Definition: Suppose $\alpha < a < b < \beta$ then if $g = (a, b)$ is a segment, define $\ell(g) = b - a$.

Definition: Suppose that $M \subset (\alpha, \beta)$ then the outer measure of M, denoted by $\mu_o(M)$, is defined as:

$$\mu_o(M) = \inf \left\{ \sum_{g \in G} \ell(g) \middle| G \text{ is a finite collection of segments covering } M \right\}.$$

Definition: Suppose that $M \subset (\alpha, \beta)$ then the inner measure of M, denoted by $\mu_i(M)$, is defined as:

$$\mu_i(M) = (\beta - \alpha) - \mu_o(\ (\alpha, \beta) - M).$$

Definition: If the outer measure of a set M is equal to the inner measure of the set; then the measure of M, denoted by $\mu(M)$, is this number:

$$\mu(M) = \mu_o(M) = \mu_i(M) = (\beta - \alpha) - \mu_o(\ (\alpha, \beta) - M).$$

Observe that this implies that if M is measurable then

$$\mu_o(M) + \mu_o(\ (\alpha, \beta) - M) \quad = \quad \beta - \alpha.$$

Exercise 6.1. Observe that the value of $\beta - \alpha$ was needed to calculate $\mu(M)$. Suppose that $(\alpha, \beta) \subset (\alpha_1, \beta_1)$ and that $\mu(M)$ is calculated using (α, β) and another measure $\mu_1(M)$ is calculated using (α_1, β_1). What is the relationship between $\mu(M)$ and $\mu_1(M)$?

Exercises 6.2. Calculate the measure of each of the following sets (Note, assume (α, β) is large enough to contain all these sets):

a.) A singleton point $\{p\}$.
b.) The interval $[0, 1]$.
c.) The set $\{\frac{1}{n} | n \in \mathbb{N}\}$.
d.) The set of rational numbers between 0 and 1.
e.) The set $\cup_{i=1}^{\infty}(\frac{1}{2i+1}, \frac{1}{2i})$.

Exercises 6.3. Show whether or not the following hold (if necessary, assume that all the sets mentioned have measure.)
a.) $\mu((\alpha, \beta) - M) = \beta - \alpha - \mu(M)$.
b.) If $A \cap B = \emptyset$ then $\mu(A \cup B) = \mu(A) + \mu(B)$.
c.) If $\mu(A \cap B) = 0$ then $\mu(A \cup B) = \mu(A) + \mu(B)$.
d.) If $\mu(M) = 0$ and $K \subset M$ then $\mu(K) = 0$.
[Note: recall that for these theorems we are assuming that all sets are subsets of (α, β).]

Some helpful lemmas:
Lemma 6a. If $G = \{g_i\}_{i=1}^{n}$ is a finite collections of disjoint segment lying in (a, b), then $\cup G$ is measurable and

$$\mu(\cup G) \quad = \quad \sum_{i=1}^{n} \ell(g_i).$$

Lemma 6b. Whether or not M is measurable, we have:

$$\mu_o(M) + \mu_o((\alpha, \beta) - M) \quad \geq \quad \beta - \alpha.$$

Theorem 6.1. Suppose that M is a set and $\mu(M) = 0$. Then if $H \subset M$ then $\mu(H) = 0$.

Theorem 6.2. Suppose that M is a countable set. Then M is measurable.

Theorem 6.2. Suppose that H and K are two sets with $H \cup K = \emptyset$. Then

$$\mu_o(H \cup K) \quad \leq \quad \mu_o(H) + \mu_o(K).$$

Question: under the hypothesis of Theorem 6.2, when is

$$\mu_o(H \cup K) \quad \neq \quad \mu_o(H) + \mu_o(K).$$

Theorem 6.3. If M is compact then M is measurable.

Corollary to Theorem 6.3. If U is open (and a subset of (α, β)) then U is measurable.

Theorem 6.4. Suppose that $\{M_n\}_{n=1}^{\infty}$ is a sequence of mutually exclusive compact set. Then $\cup_{n=1}^{\infty} M_n$ is measurable and

$$\mu\Big(\cup_{n=1}^{\infty} M_n \Big) \quad = \quad \sum_{n=1}^{\infty} \mu(M_n).$$

Theorem 6.5. Suppose that $M \subset (\alpha, \beta)$. Then M is measurable if and only if it is true that if $\epsilon > 0$ there exists a collection G_M of non-overlapping segments covering M and a collection G_{I-M} of non-overlapping segments covering $I - M$ such that the sum of the lengths of all the overlaps of the segments of G_M with the segments of G_{I-M} is less than ϵ.

Theorem 6.6. If each of H and K are measurable, then so are $H \cup K$ and $H \cap K$.

Theorem 6.7. If each of H and K is measurable then:

$$\mu(K - H) \quad = \quad \mu(K) - \mu(H).$$

Theorem 6.8. If for each $\{M_i\}_{i=1}^{\infty}\}$ is a sequence of mutually disjoint measurable sets then $\cup_{i=1}^{\infty} M_i$ is measurable and

$$\mu\Big(\bigcup_{i=1}^{\infty} M_i \Big) \quad = \quad \sum_{i=1}^{\infty} \mu(M_i).$$

Exercise: Is every set measurable?