More Integrability Exercises

Exercise 5. Show that the function defined in Exercise 2.1 e is not integrable. [Hint: show that for any number I there and any $\delta > 0$ that there is a subdivision S with mesh$(S) < \delta$ so that

$$\sum_{i=1}^{n} f(x_i^*)(x_i - x_{i-1}) > I.$$

And indicate why this is sufficient.]

Exercise 6. Show that if f is continuous over the interval $[a, b]$ then if $\epsilon > 0$ there exists a $\delta > 0$ so that if S is a subdivision with mesh$(S) < \delta$ then:

$$\text{Upper Sum}(S, f) - \text{Lower Sum}(S, f) < \epsilon$$

Exercise 7. Repeat Exercise 6 but with a (possibly discontinuous) increasing function define on the interval $[a, b]$.

Exercise 8.
Definition: If $S : \{a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b\}$ is a subdivision then the subdivision $T : \{a = t_0 < t_1 < t_2 < \ldots < t_{m-1} < x_m = b\}$ is a refinement of S means that for each $i, 0 \leq i \leq n$ there is an integer j so that $t_j = x_i$.

Show that if the subdivision $T : \{a = t_0 < t_1 < t_2 < \ldots < t_{m-1} < x_m = b\}$ is a refinement of the subdivision $S : \{a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b\}$, then

$$\text{Lower Sum}(S, f) \leq \text{Lower Sum}(T, f) \leq \text{Upper Sum}(T, f) \leq \text{Upper Sum}(S, f).$$