Proof Of Integrability, Exercise 2.1d

Exercise: Let f be defined as follows:

$$f(x) = \begin{cases} 1 & \text{if } 0 \leq x < 1 \\ 3 & \text{if } 1 \leq x \leq 2 \end{cases}$$

In order to prove integrability we need to find a number I so that if $\varepsilon > 0$ then there exists $\delta > 0$ so that for a subdivision S of mesh less than δ we have

$$\Big| \sum_{i=1}^{n} f(x_i^*)(x_i - x_{i-1}) - I \Big| \quad < \quad \varepsilon \tag{1}$$

where S is the subdivision $\{0 = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = 2\}$ and for each i, $x_i^* \in [x_{i-1}, x_i]$. Our plan is as follows: split the sum into three pieces: (1) the sum over the intervals up to the one containing 1, (2) the interval containing 1, (3) the sum over the intervals after the one containing 1.

So here's the proof:

Proof. First we need to calculate I, a quick examination (from what we learned in calculus) allows us to make the educated guess that $I = 4$. Let $\varepsilon > 0$ and (based on the scratch work done in class) we let δ be a positive number with $\delta < \frac{\varepsilon}{9}$. To complete the proof we will show that if S is the subdivision $\{0 = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = 2\}$ with mesh less than δ then equation (1) above holds. Suppose, then, that S is a subdivision of $[0, 2]$ with mesh less than δ. Let k be the first integer so that $1 \in [x_{k-1}, x_k]$. (Note, I phrase it this way because there is a possibility that for some i, $x_i = 1$; in that case 1 is in both $[x_{i-1}, x_i]$ and $[x_i, x_{i+1}]$. If $f(1)$ had been 1 instead of 3 I would have picked k to be the largest integer such that $1 \in [x_{k-1}, x_k]$.) So we will have:

$$\Big| \sum_{i=1}^{n} f(x_i^*)(x_i - x_{i-1}) - 4 \Big| =$$

$$\left|\sum_{i=1}^{k-1} f(x_i^*)(x_i - x_{i-1}) + f(x_k^*)(x_k - x_{k-1}) + \sum_{i=k+1}^{n} f(x_i^*)(x_i - x_{i-1}) - 4\right| =$$

$$\left|\sum_{i=1}^{k-1} f(x_i^*)(x_i - x_{i-1}) - 1 + f(x_k^*)(x_k - x_{k-1}) + \sum_{i=k+1}^{n} f(x_i^*)(x_i - x_{i-1}) - 3\right| \leq$$

$$\left|\sum_{i=1}^{k-1} f(x_i^*)(x_i - x_{i-1}) - 1\right| + \left|f(x_k^*)(x_k - x_{k-1})\right| + \left|\sum_{i=k+1}^{n} f(x_i^*)(x_i - x_{i-1}) - 3\right|.$$

Let's look at each of these absolute values separately: The first one follows, and note that in this case $f(x_i^*) = 1$ for all $1 \leq i \leq k-1$:

$$\begin{aligned}
\left|\sum_{i=1}^{k-1} f(x_i^*)(x_i - x_{i-1}) - 1\right| &= \Big|f(x_1^*)(x_1 - x_0) + f(x_2^*)(x_2 - x_1) + \dots \\
&+ f(x_{k-2}^*)(x_{k-2} - x_{k-3}) + f(x_{k-1}^*)(x_{k-1} - x_{k-2}) - 1\Big| \\
&= \Big|1(x_1 - 0) + 1(x_2 - x_1) + \dots \\
&+ 1(x_{k-2} - x_{k-3}) + 1(x_{k-1} - x_{k-2}) - 1\Big| \\
&= \Big|x_{k-1} - 1\Big| \leq \text{ mesh}(S) < \delta < \frac{\varepsilon}{9}.
\end{aligned}$$

The last line follows from the fact that $1 \in [x_{k-1}, x_k]$ and $x_k - x_{k-1} \leq \text{mesh}(S) < \delta$.

The second term gives us:

$$\left|f(x_k^*)(x_k - x_{k-1})\right| \leq 3(x_k - x_{k-1}) \leq 3\text{mesh}(S) < 3\delta < 3\frac{\varepsilon}{9} = \frac{\varepsilon}{3}.$$

The third term gives us:

$$
\begin{aligned}
\Big|\sum_{k+1}^{n} f(x_i^*)(x_i - x_{i-1}) - 3\Big| &= \Big|f(x_{k+1}^*)(x_{k+1} - x_k) + f(x_{k+2}^*)(x_{k+2} - x_{k+1}) + \dots \\
&+ f(x_{n-1}^*)(x_{n-1} - x_{n-2}) + f(x_n^*)(x_n - x_{n-1}) - 3\Big| \\
&= \Big|3(x_{k+1} - x_k) + 3(x_{k+2} - x_{n+1}) + \dots \\
&+ 3(x_{n-1} - x_{n-2}) + 3(x_n - x_{n-1}) - 3\Big| \\
&= \Big| - 3x_k + 3x_n - 3\Big| \\
&= \Big| - 3x_k + 3 - 3 + 3x_n - 3\Big| \\
&= \Big| - 3x_k + 3 - 3 + 3 \cdot 2 - 3\Big| \\
&= \Big| - 3x_k + 3| \\
&= \Big|3(1 - x_k)| \\
&\leq 3 \text{ mesh}(S) < 3\delta < 3\frac{\varepsilon}{9} = \frac{\varepsilon}{3}.
\end{aligned}
$$

Therefore:

$$
\text{first term} + \text{second term} + \text{third term} \; < \; \frac{\varepsilon}{9} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} < \varepsilon.
$$

Which is what was needed. □

Exercises

Exercises: In each of the following exercises, show from the definition that the indicated function is integrable.

Exercise 1. Suppose that $a < c < b$ and e and d are positive numbers. Define

$$
f(x) = \begin{cases} e & \text{if } a \leq x < b \text{ and } x \neq c \\ d & \text{if } x = c. \end{cases}
$$

Exercise 2. Define

$$f(x) = 5x.$$

Show, from the definition, that f is integrable over the interval $[2, 5]$.
As a hint do the following:

Suppose that you are given a subdivision S of the interval $[2, 5]$. Then what is the possible range of values for different choices of x_i^* for a subdivision of mesh δ. Fist consider the case where all the intervals are the same length: $x_i - x_{i-1} = \delta$.

Repeat the above in the case where the intervals of the subdivision have different lengths.

Exercise 3. Suppose that $a < c < b$ and e and d is a positive number. Define

$$f(x) = \begin{cases} 5x & \text{if } a \leq x < b \text{ and } x \neq c \\ d & \text{if } x = c. \end{cases}$$

Exercise 4. Potential lemma:
Suppose that S is a subdivision with $S : \{a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b\}$. If f is a function, for this particular subdivision and function f, then define for each i,

$$\begin{aligned} y_i^m &= \operatorname{glb}\{f(x_i^*) \mid x_i^* \in [x_{i-1}, x_i]\} \\ y_i^M &= \operatorname{lub}\{f(x_i^*) \mid x_i^* \in [x_{i-1}, x_i]\}. \end{aligned}$$

Then for the function f we define

$$\begin{aligned} \text{Lower Sum}(S, f) &= \sum_{i=1}^{n} y_i^m (x_i - x_{i-1}) \\ \text{Upper Sum}(S, f) &= \sum_{i=1}^{n} y_i^M (x_i - x_{i-1}). \end{aligned}$$

Consider the function $f(x) = cx$ for some constant $c > 0$; argue that if $a < b$ then if $\varepsilon > 0$ there exists a number $\delta > 0$ so that if S is a subdivision of $[a, b]$ with $\text{mesh}(S) < \delta$ then

$$\text{Upper Sum}(S, f) - \text{Upper Sum}(S, f) < \varepsilon. \tag{2}$$

Is this enough to guarantee that f is integrable over $[a, b]$?