Quiz02.

I just discovered that I labeled this page Quiz02 rather than Quiz01. I want to alert you that this is the first quiz for this semester; so don't worry if you've not turned in two quizzes.

The quiz is due before by class time, 12:30 pm February 02. Quiz may be handwritten and should be converted to a pdf file which is to be emailed to me with the file name beginning with your last name.

Make sure to show all your work. You may not receive full credit if the accompanying work is incomplete or incorrect. If you do scratch work make sure to indicate scratch work - I will not take off points for errors in the scratch work if it is so labeled. Make sure to distinguish between scratch work and proof.

Note that all the proofs must follow logically from the theorems and definitions stated in the class notes; if you wish to use some lemma that has not be proven in class, you must prove it first using the theorems and definitions stated from the class notes.

In all these problems assume that any require quantities exist. (E.g. if I have $f'(p)$ as part of a formula, then assume that f is a function differentiable at p.)

Problem 1. Suppose:

$$\lim_{x\to p} f(x) = q \quad \text{and} \quad \lim_{x\to q} g(x) = r.$$

Then use the $\epsilon - \delta$ definition of limit to show that

$$\lim_{x\to p} g(f(x)) \;\; = \;\; r.$$

Proof. Let $\epsilon > 0$. Since $\lim_{x\to q} g(x) = r$ there exists a number $\delta_1 > 0$ so that if $0 < |x - q| < \delta_1$ then

$$|g(x) - r| \;\; < \;\; \epsilon.$$

Since $\lim_{x\to p} f(x) = q$ there exists a number $\delta_2 > 0$ so that if $0 < |t - p| < \delta_2$ then

$$|f(t) - q| \;\; < \;\; \delta_1.$$

Therefore, if $0 < |t-p| < \delta_2$ we have

$$|f(t) - q| \quad < \quad \delta_1.$$

Then by letting $f(t)$ take on the role of x in the first equations, we have:

$$|g(f(t)) - r| \quad < \quad \epsilon.$$

□

Problem 2. Suppose $f : \mathbb{R} \to \mathbb{R}$ and that for all $x \in \mathbb{R}$ we have:

$$-x^2 \leq f(x) \leq x^2.$$

Use the $\epsilon - \delta$ definition of limit to prove that $f'(0) = 0$.

Proof. Observe that the above equation is equivalent to $|f(x)| < x^2$; and note that this implies that $f(0) = 0$.

We need to show that

$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h} \quad = \quad 0.$$

Since $f(0) = 0$ we need to show that

$$\lim_{h \to 0} \frac{f(h)}{h} \quad = \quad 0.$$

Toward that end, let $\epsilon > 0$ then let $\delta = \epsilon$; then if $0 < |h| < \delta$ we have

$$\begin{aligned}
|h| &< \epsilon \\
\left|\frac{h^2}{h}\right| &< \epsilon \\
\left|\frac{f(h)}{h}\right| &\leq \left|\frac{h^2}{h}\right| \\
\therefore \left|\frac{f(h)}{h}\right| &< \epsilon
\end{aligned}$$

which is what was needed.

□

Problem 3. Prove the Lemma to Theorem 1.5:

If $f : [a, b] \to \mathbb{R}$ is a function, $a < p < b$ and $f'(p) \neq 0$, then if $\delta > 0$ there is a point $q \in (p - \delta, p + \delta)$ so that $f(q) > f(p)$.

Proof. Suppose $f'(p) \neq 0$.

Case 1: $f'(p) > 0$. Select $\epsilon = (f'(p))/2$. Then, from the limit definition, let $\epsilon = (f'(p))/2$, and we have a $\delta > 0$ so that if $0 < |x - p| < \delta$,

$$\left|\frac{f(p+h) - f(p)}{h} - f'(p)\right| \quad < \quad \epsilon = \frac{f'(p)}{2}.$$

So this gives us

$$\begin{aligned} -\frac{f'(p)}{2} < & \quad \tfrac{f(p+h)-f(p)}{h} - f'(p) & < \frac{f'(p)}{2} \\ \frac{f'(p)}{2} < & \quad \tfrac{f(p+h)-f(p)}{h} & < \frac{3f'(p)}{2}. \end{aligned}$$

Select $0 < h < \delta$ then

$$h\frac{f'(p)}{2} < \quad f(p+h) - f(p) \quad < h\frac{3f'(p)}{2}.$$

Then the left term in the line above is positive, so

$$\begin{aligned} 0 < h\frac{f'(p)}{2} & < & f(p+h) - f(p) \\ 0 & < & f(p+h) - f(p) \\ f(p) & < & f(p+h). \end{aligned}$$

So $q = p + h$ is a point so that $f(q) > f(p)$.

[Note the case for $f'(p)$ being negative is enough different that it needs to be done as well; it uses the fact that h can be negative.]

Case 2: $f'(p) < 0$. Select $\epsilon = -(f'(p))/2$. Again, from the limit definition, if we let $\epsilon = -(f'(p))/2$, we have a $\delta > 0$ so that if $0 < |x - p| < \delta$,

$$\left|\frac{f(p+h) - f(p)}{h} - f'(p)\right| \quad < \quad \epsilon = -\frac{f'(p)}{2}.$$

So this gives us

$$\begin{aligned}
\frac{f'(p)}{2} < &\quad \tfrac{f(p+h)-f(p)}{h} - f'(p) &< -\frac{f'(p)}{2} \\
\frac{3f'(p)}{2} < &\quad \tfrac{f(p+h)-f(p)}{h} &< \frac{f'(p)}{2}
\end{aligned}$$

Select $h < 0$ so that $0 < -h < \delta$ then (since h is negative the inequalities switch),

$$h\frac{3f'(p)}{2} > \quad f(p+h) - f(p) \quad > h\frac{f'(p)}{2}.$$

Then the right term in the line above is a negative times a negative, so it's positive and this gives

$$\begin{aligned}
f(p+h) - f(p) &> 0 \\
f(p+h) &> f(p).
\end{aligned}$$

So $q = p + h$ is a point so that $f(q) > f(p)$. $\square$

Note that in the proof details we see that if $f'(p)$ is positive then q is on the right of p and if $f'(p)$ is negative then q is on the left of p.