Math 5210 Quiz 02 Spring 2021.

The quiz is due before class Tuesday March 2. The quiz is open notes. You may not receive any other outside assistance and may not discuss the test with anyone. Show all your work, you may not receive full credit if the accompanying work is incomplete or incorrect.

For these problems you may assume any of the theorems in 0.x - 3.x notes. Make sure to indicate when you use one of these theorems in your work.

You will be graded on the best three out of the four. [Equivalently: you may skip one of the problems, but I'd like you to try all four.]

Problem 1. Consider the following function:

$$f(x) = \begin{cases} x & \text{if } 0 \leq x \leq 1 \\ 2x & \text{if } 1 < x \leq 2. \end{cases}$$

Define the function $F(t)$ as follows:

$$F(t) = \int_0^t f(x)dx.$$

a.) Derive the formula for $F(x)$.

Solution. Sine the function f is increasing we know that it is integrable. So I'll use Theorem 3.1 to obtain the integral:

$$\int_{[a,b]} f = \lim_{n\to\infty} \sum_{i=1}^{n} f\Big(a + i\frac{b-a}{n}\Big)\frac{b-a}{n}.$$

So, for $t \leq 1$:

$$\begin{aligned}
\int_a^t f(x)dx &= \lim_{n\to\infty} \sum_{i=1}^n f\Big(0 + i\frac{t}{n}\Big)\frac{t}{n} \\
&= \lim_{n\to\infty} \sum_{i=1}^n \Big(i\frac{t}{n}\Big)\frac{t}{n} \\
&= \lim_{n\to\infty} \frac{t^2}{n^2}\sum_{i=1}^n i \\
&= \lim_{n\to\infty} \frac{t^2}{n^2}\frac{n(n+1)}{2} \\
&= t^2 \lim_{n\to\infty} \frac{n(n+1)}{2n^2} \\
&= t^2\frac{1}{2} = \frac{t^2}{2}.
\end{aligned}$$

For $1 < t$ we first calculate $\int_1^t 2xdx$

$$\begin{aligned}
\int_1^t 2xdx &= \lim_{n\to\infty} \sum_{i=1}^n f\Big(1 + i\frac{t-1}{n}\Big)\frac{t-1}{n} \\
&= \lim_{n\to\infty} \sum_{i=1}^n 2\Big(1 + i\frac{t-1}{n}\Big)\frac{t-1}{n} \\
&= 2\lim_{n\to\infty} \Big(\sum_{i=1}^n \frac{t-1}{n} + \sum_{i=1}^n i\frac{(t-1)^2}{n^2}\Big) \\
&= 2\lim_{n\to\infty} \Big(\frac{t-1}{n}\sum_{i=1}^n 1 + \frac{(t-1)^2}{n^2}\sum_{i=1}^n i\Big) \\
&= 2\lim_{n\to\infty} \Big(\frac{t-1}{n}n + \frac{(t-1)^2}{n^2}\frac{n(n+1)}{2}\Big) \\
&= 2(t-1) + 2\frac{(t-1)^2}{2} \\
&= 2t - 2 + t^2 - 2t + 1 \\
&= t^2 - 1.
\end{aligned}$$

Then we use the fact that $\int_a^c + \int_c^b = \int_a^b$ for calculating F for $t > 1$:

$$
\begin{aligned}
F(t) &= \int_0^t f(x)dx \\
&= \int_0^1 f(x)dx + \int_0^t f(x)dx \\
&= \int_0^1 xdx + \int_0^t 2xdx \\
&= \frac{1}{2} + t^2 - 1 \\
&= t^2 - \frac{1}{2}.
\end{aligned}
$$

Therefore:

$$
F(t) = \begin{cases} \frac{t^2}{2} & \text{if } 0 \le x \le 1 \\ t^2 - \frac{1}{2} & \text{if } 1 < x \le 2. \end{cases}
$$

[Note: something should be said about the validity of this formula given the discontinuity at $x = 1$. I won't insist on that, but a comment that says something about what we've discovered about the value of the integral of a bounded function discontinuous at only one point would be appropriate. You likely recall from class that the value of f at the point of discontinuity does not affect the value of the integral.] □

b.) Prove that F is differentiable at $x = 1.5$ and calculate $F'(1.5)$.

Solution.

$$
\begin{aligned}
F'(1.5) &= \lim_{h \to 0} \frac{F(1.5+h) - F(1.5)}{h} \\
&= \lim_{h \to 0} \frac{((1.5+h)^2 - \frac{1}{2}) - ((1.5)^2 - \frac{1}{2})}{h} \\
&= \lim_{h \to 0} \frac{(1.5^2 + 3h + h^2) - (1.5)^2}{h} \\
&= \lim_{h \to 0} \frac{3h + h^2}{h} \\
&= \lim_{h \to 0} 3 + h \\
&= 3.
\end{aligned}
$$

□

c.) Prove that F is not differentiable at $x = 1$.

Solution. It is sufficient to show that the limits as $h \to 0$ is different for negative h than for positive h; equivalently: that the limit from the left is different from the limit from the right.

$$\begin{aligned}
F'(1) &= \lim_{h \to 0^-} \frac{F(1+h) - F(1)}{h} \\
&= \lim_{h \to 0^-} \frac{\frac{(1+h)^2}{2} - \frac{1}{2}}{h} \\
&= \lim_{h \to 0^-} \frac{h + \frac{h^2}{2}}{h} \\
&= 1 \\
F'(1) &= \lim_{h \to 0^+} \frac{F(1+h) - F(1)}{h} \\
&= \lim_{h \to 0^+} \frac{(1+h)^2 - \frac{1}{2} - \frac{1}{2}}{h} \\
&= \lim_{h \to 0^+} \frac{1 + 2h + h^2 - 1}{h} \\
&= \lim_{h \to 0^+} 2 + h \\
&= 2.
\end{aligned}$$

Since $1 \neq 2$ we have a contradiction so the limit does not exist and so F does not have a derivative at $x = 1$.

□

2. Suppose that f is an increasing function on the interval $[a, b]$. Let

$$F(t) = \int_a^t f(x)dx.$$

a.) Argue that if $a < x^* < b$ then

$$f(a) < f(x^*) < f(b).$$

Solution. This follows from the fact that the function is strictly increasing.

□

b.) Prove that F is continuous over $[a, b]$.

Solution. We've proven in class that if $g(x) \leq h(x)$ for all $x \in [p, q]$ then $\int_p^q g(x)dx \leq \int_p^q h(x)dx$. Observe that this implies that $|\int_p^q f(x)dx| \leq \int_p^q |f(x)|dx$. We also know that an increasing function is bounded. [It's bounded by the larger of $f(b)$ or $|f(a)|$.] Let B be a bound of f such that $|f(x)| < B$ for all $x \in [a, b]$.

Let $\epsilon > 0$ and select $\delta > 0$ so that $\delta < \frac{\epsilon}{B}$. Then if $|q - p| < \delta$ (we can assume $p < q$) we have:

$$\begin{aligned}
|F(q) - F(p)| &= \Big| \int_a^q f(x)dx - \int_a^p f(x)dx \Big| \\
&= \Big| \int_p^q f(x)dx \Big| \\
&\leq \int_p^q |f(x)|dx \\
&\leq \int_p^q B dx \\
&\leq B(q - p) \\
&< B\delta \\
&< B\frac{\epsilon}{B} = \epsilon.
\end{aligned}$$

Observe that we have proven that the function is uniformly continuous. □

3. Suppose that $f : [a, b] \to R$ is a function; for each k, $S_k = \{a = x_{k,0} < x_{k,1} < x_{k,2} < \cdots < x_{k,n_k-1} < x_{k,n_k} = b\}$ is a subdivision of $[a, b]$ and that $\text{mesh}(S_k) < \frac{1}{k}$; and let $x^*_{k,i} \in [x_{k,i-1}, x_{k,i}]$. Define

$$I_k = \sum_{i=1}^{n_k} f(x^*_{k,i})(x_{k,i} - x_{k,i-1}).$$

Show that if f is integrable over $[a, b]$ then the sequence $\{I_k\}_{k=1}^{\infty}$ converges to the integral $\int_a^b f$.

Solution. Let $\epsilon > 0$. Then, since f there exists a number $\delta > 0$ so that if S is a subdivision of $[a, b]$ with $\text{mesh}(S) < \delta$ then

$$\Big| \sum_{i=1}^{n} f(x^*_i)(x_i - x_{i-1}) - \int_a^b f(x)dx \Big| \quad < \quad \epsilon$$

then select $N \in \mathbb{N}$ so that $N > \frac{1}{\delta}$. Then if $k > N$ we have $\frac{1}{k} < \delta$ so S_k is a subdivision of $[a, b]$ with $\text{mesh}(S_k) < \frac{1}{k} < \delta$. So, no matter what selection is used for x^* we have

$$\begin{aligned} \Big| \sum_{i=1}^{n_k} f(x_{k,i}^*)(x_{k,i} - x_{k,i-1}) - \int_a^b f(x)dx \Big| &< \epsilon \\ |I_k - \int_a^b f(x)dx| &< \epsilon. \end{aligned}$$

Therefore, the sequence $\{I_k\}_{k=1}^{\infty}$ converges to $\int_a^b f(x)dx$.

□

4. For each integer $n \in \mathbb{N}$ let

$$f_n(x) = \left(x + \frac{1}{n}\right)^2.$$

a.) Determine the function to which the sequence $\{f_n\}_{n=1}^{\infty}$ converges.

Solution. The functions $\{f_n\}_{n=1}^{\infty}$ converge to the function $f(x) = x^2$. □

b.) Prove that it converges uniformly to that function on the interval $[1, 2]$.

Solution. Let $\epsilon > 0$. Select $N \in \mathbb{N}$ so that:

$$\begin{aligned} N &> \sqrt{\frac{2}{\epsilon}} \\ N &> \frac{8}{\epsilon}. \end{aligned}$$

Observe that this implies that if $n > N$, then

$$\begin{aligned} \frac{1}{n^2} &< \frac{\epsilon}{2} \\ \frac{4}{n} &< \frac{\epsilon}{2}. \end{aligned}$$

[I omitted the scratch work, but this observation should indicate where it comes from later in the proof.]

Then, if $x \in [1, 2]$ we have $1 \le x \le 2$ so that:

$$\begin{aligned}
|f_n(x) - f(x)| &= \left|\left(x + \frac{1}{n}\right)^2 - x^2\right| \\
&= \left|x^2 + 2x\frac{1}{n} + \frac{1}{n^2} - x^2\right| \\
&= \left|2x\frac{1}{n} + \frac{1}{n^2}\right| \\
&\le \left|2x\frac{1}{n}\right| + \left|\frac{1}{n^2}\right| \\
&\le \left|4\frac{1}{n}\right| + \left|\frac{1}{n^2}\right| \\
&< 4\frac{1}{n} + \frac{1}{n^2} \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

□