Math 5210 Quiz 02 Spring 2021.

The quiz is due before class Tuesday March 2. The quiz is open notes. You may not receive any other outside assistance and may not discuss the test with anyone. Show all your work, you may not receive full credit if the accompanying work is incomplete or incorrect.

For these problems you may assume any of the theorems in 0.x - 3.x notes. Make sure to indicate when you use one of these theorems in your work.

You will be graded on the best three out of the four. [Equivalently: you may skip one of the problems, but I'd like you to try all four.]

Problem 1. Consider the following function:

$$f(x) = \begin{cases} x & \text{if } 0 \leq x \leq 1 \\ 2x & \text{if } 1 < x \leq 2. \end{cases}$$

Define the function $F(t)$ as follows:

$$F(t) = \int_0^t f(x)dx.$$

a.) Derive the formula for $F(x)$.
b.) Prove that F is differentiable at $x = 1.5$ and calculate $F'(1.5)$.
c.) Prove that F is not differentiable at $x = 1$.

2. Suppose that f is an increasing function on the interval $[a, b]$. Let

$$F(t) = \int_a^t f(x)dx.$$

a.) Argue that if $a < x^* < b$ then

$$f(a) < f(x^*) < f(b).$$

b.) Prove that F is continuous over $[a, b]$.

3. Suppose that $f : [a, b] \to R$ is a function; for each k, $S_k = \{a = x_{k,0} < x_{k,1} < x_{k,2} < \cdots < x_{k,n_k-1} < x_{k,n_k} = b\}$ is a subdivision of $[a, b]$ and that $\text{mesh}(S_k) < \frac{1}{k}$; and let $x^*_{k,i} \in [x_{k,i-1}, x_{k,i}]$. Define

$$I_k = \sum_{i=1}^{n_k} f(x^*_{k,i})(x_{k,i} - x_{k,i-1}).$$

Show that if f is integrable over $[a, b]$ then the sequence $\{I_k\}_{k=1}^{\infty}$ converges to the integral $\int_a^b f$.

4. For each integer $n \in \mathbb{N}$ let

$$f_n(x) \quad = \quad \left(x + \frac{1}{n}\right)^2.$$

a.) Determine the function to which the sequence $\{f_n\}_{n=1}^{\infty}$ converges.

b.) Prove that it converges uniformly to that function on the interval $[1, 2]$.