Test 1, Math 5210/6210
Dr. Smith

The test is due before class Tuesday March 23. The test is open notes and open book. ***You may not receive any other outside assistance and may not discuss the test with anyone.*** Show all your work, you may not receive full credit if the accompanying work is incomplete or incorrect.

If you are asked to prove something from the definition or from the $\epsilon - \delta$ definition then make sure to use the appropriate definition. You are allowed to use the "short cut" methods from calculus in your scratch work; but in order to get full credit on a problem, you must use the indicated definition.

In proving a theorem (e.g. problem 3) you may use the theorems that precede it in the notes and any of our theorems from 5210/6210.

Problem 1. Consider the function

$$f(x) = \begin{cases} x^2 & \text{if } x \text{ is rational} \\ -x^2 & \text{if } x \text{ is irrational} \end{cases}$$

(a) Use the $\epsilon - \delta$ definition of the limit to prove that f is differentiable at $x = 0$.

(b) Prove that f is not continuous at all points with $x \neq 0$.

Problem 2. For each $n \in \mathbb{Z}$ let $f_n(x) = x^n$ for all x where $f_n(x)$ is defined.

(a) Use the limit definition to calculate $f_2'(x)$.

(b) For each $n \in \mathbb{Z}$ derive the formula for $f_n'(x)$. [Hint: use induction and the results of theorem 1.3. And don't forget negative n's!]

Problem 3. Prove lemma 2.2b. Caution: make sure to do both directions of the *if and only if* statement. In additions to theorems preceding the lemma, you are allowed to use some of the results obtained in Quiz 02.

Problem 4. Let $f(x) = \sqrt{x+1}$ for $x > 0$. Calculate $f'(3)$ and use the $\epsilon - \delta$ definition of the limit to prove that your calculation is correct.

Problem 5. For each positive integer n let

$$f_n(x) = \sqrt{x + \frac{x}{n}}.$$

Prove that the sequence $\{f_n\}_{n=1}^{\infty}$ converges uniformly over the interval $[2, 5]$.

Extra credit: Prove that the sequence converges uniformly over the interval $[0, b]$ for every $b > 0$.

Problem 6. Consider the function g where

$$g(x) = \begin{cases} 0 & \text{if } x < 2 \\ 2 & \text{if } 2 \leq x \leq 3 \\ 5 & \text{if } 3 < x \end{cases}$$

For each of the following functions f, determine the value of $\int_0^5 f dg$. Then use the $\epsilon - \delta$ definition of the integral to prove that your value is correct.

$$\begin{array}{lll} \text{(a)} & f(x) = & 4x^2 \\ \text{(b)} & f(x) = & \sqrt{x}. \end{array}$$

Problem 7.

(a) Prove that if f is an increasing function then the variation function satisfies:

$$\mathcal{V}_a^x f \quad = \quad f(x) - f(a).$$

(b) Given the function $f(x) = x(x-2)(x-3)$. Calculate the function $V(x) = \mathcal{V}_{-1}^x f$ and sketch its graph.

Extra credit problem:

(a) Prove that the union of a finite number of nowhere dense sets is nowhere dense. [Hint: use induction.]

(b) Prove that if the set M has a finite number of limit points then it is nowhere dense.