Test 1, Math 5210/6210
Dr. Smith

The test is due before class Tuesday March 23. The test is open notes and open book. ***You may not receive any other outside assistance and may not discuss the test with anyone.*** Show all your work, you may not receive full credit if the accompanying work is incomplete or incorrect.

If you are asked to prove something from the definition or from the $\epsilon - \delta$ definition then make sure to use the appropriate definition. You are allowed to use the "short cut" methods from calculus in your scratch work; but in order to get full credit on a problem, you must use the indicated definition.

In proving a theorem (e.g. problem 3) you may use the theorems that precede it in the notes and any of our theorems from 5210/6210.

Problem 1. Consider the function

$$f(x) = \begin{cases} x^2 & \text{if } x \text{ is rational} \\ -x^2 & \text{if } x \text{ is irrational} \end{cases}$$

(a) Use the $\epsilon - \delta$ definition of the limit to prove that f is differentiable at $x = 0$.

Proof. Observe that $|f(x)| = x^2$ for all x.

We claim that the derivative at $x = 0$ is 0. So we will prove that the limit is 0. Toward that end: Let $\epsilon > 0$. Let $\delta < \epsilon$. Then let $0 < |h| < \delta$. If h is rational we have:

$$\begin{aligned} |h| &< \epsilon \\ \left|\frac{h^2}{h}\right| &< \epsilon \\ \left|\frac{(0+h)^2 - 0}{h} - 0\right| &< \epsilon \\ \left|\frac{f(0+h) - f(0)}{h} - 0\right| &< \epsilon. \end{aligned}$$

If h is irrational we have:

$$\begin{aligned}
|-h| &< \epsilon \\
\Big|\frac{-h^2}{h}\Big| &< \epsilon \\
\Big|\frac{-(0+h)^2-0}{h}-0\Big| &< \epsilon \\
\Big|\frac{f(0+h)-f(0)}{h}-0\Big| &< \epsilon.
\end{aligned}$$

Together this gives us

$$\Big|\frac{f(0+h)-f(0)}{h}-0\Big| < \epsilon,$$

so

$$\lim_{h\to 0}\Big(\frac{f(0+h)-f(0)}{h}\Big) = 0.$$

This means $f'(0) = 0$. □

(b) Prove that f is not continuous at all points with $x \neq 0$.

Proof. We will prove that if $p \neq 0$ then f is not continuous at p. So we let $\epsilon = |p|^2$. Let $\delta > 0$. First we consider the case where p is positive. Then there is a rational number $r > p$ with $p < r < p+\delta$ and there is an irrational number $s > p$ with $p < r < p + \delta$.

Case 1. p is rational:
Then

$$\begin{aligned}
|f(p)-f(s)| &= |p^2 - -s^2| \\
&= p^2 + s^2 \\
&> 2p^2 > \epsilon.
\end{aligned}$$

Case 2. p is irrational:
Then

$$\begin{aligned}
|f(p)-f(r)| &= |-p^2 - r^2| \\
&= p^2 + r^2 \\
&> 2p^2 > \epsilon.
\end{aligned}$$

For the case where p is negative, there is a rational number $r < p$ with $p - \delta < r < p$ and there is an irrational number $s < p$ with $p - \delta < s < p$. And the argument continues as in cases 1 and 2 above with attention to the signs.

In either case for the given ϵ we can always find a number q within δ of p so that $|f(p) - f(q)| > \epsilon$. □

Problem 2. For each $n \in \mathbb{Z}$ let $f_n(x) = x^n$ for all x where $f_n(x)$ is defined.

(a) Use the limit definition to calculate $f_2'(x)$.

Solution.

$$\begin{aligned}
\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} \\
&= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} \\
&= \lim_{h\to 0} \frac{2xh + h^2}{h} \\
&= \lim_{h\to 0} 2x + h \\
&= 2x.
\end{aligned}$$

□

(b) For each $n \in \mathbb{Z}$ derive the formula for $f_n'(x)$. [Hint: use induction and the results of theorem 1.3. And don't forget negative n's!]

Solution. Let $g(x) = x$. First we use the limit definition to prove that $g'(x) = 1$; this is easy. And we just proved that for $f(x) = x^2$ that $f'(x) = 2x$. Suppose that for $n \geq 2$ that we assume, inductively, that if $f(x) = x^n$ then $f'(x) = nx^{n-1}$. Then if we let $h(x) = x^{n+1}$ using formula (4) of theorem 1.3:

$$\begin{aligned}
h(x) &= x^{n+1} \\
&= x \cdot x^n \\
&= g(x) \cdot f(x) \\
h'(x) &= g'(x)f(x) + g(x)f'(x) \\
&= 1 \cdot x^n + x \cdot nx^{n-1} \\
&= (n+1)x^n.
\end{aligned}$$

And the result follows by induction.

For $m = -n$ with $n \in N$ let $f = x^n$ so if $h = \frac{1}{f}$ using formula (5) of theorem 1.3 we have:

$$\begin{aligned} h(x) &= x^m = x^{-n} \\ &= \frac{1}{x^n} \\ &= \frac{1}{f(x)} \\ h'(x) &= -\frac{f'(x)}{(f(x))^2} \\ &= -\frac{nx^{n-1}}{x^{2n}} \\ &= -nx^{-n-1} \\ &= mx^{m-1}. \end{aligned}$$

Since 0 is an integer we need to worry about the case $f(x) = x^0$. When I made out the test, it seemed obvious to me that in this case $f(x) = 1$ and so $f'(x) = 0$. But, after some thought, I realized that f is undefined at $x = 0$ since 0^0 is an undefined quantity. But the only value that makes the function f continuous is assuming $0^0 = 1$ so, in this case $x^0 = f(x) = 1$ would be differentiable and give $f'(0) = 0$. In any case, saying that $f'(x) = 0$ will be accepted as a correct solution. □

Problem 3. Prove lemma 2.2b. Caution: make sure to do both directions of the *if and only if* statement. In additions to theorems preceding the lemma, you are allowed to use some of the results obtained in Quiz 02.

Proof Outline. If f is R-integrable, then the condition follows easily from the definition of the integral and the properties of the upper and lower sums discussed in class.

So, for the other (the harder) direction of the iff statement, suppose that the condition is satisfied. The for each integer n let I_n denote some element of the set

$$\left\{ \sum_{i=1}^{m} f(x_i^*)(x_i - x_{i-1}) \middle| S : \{a = x_0 < x_1 < \ldots < x_m = b\};\ \text{mesh}(S) < \frac{1}{n} \right\}.$$

Then argue that the set $\{I_n\}_{n=1}^{\infty}$ is bounded.

This implies that some subsequence has a sequential limit point.

Argue that every subsequence must have the same limit point.

Then use the properties of upper and lower sums and a straightforward $\epsilon - \delta$ argument to show that this limit must be the R-integral of the function f. □

Problem 4. Let $f(x) = \sqrt{x+1}$ for $x > 0$. Calculate $f'(3)$ and use the $\epsilon - \delta$ definition of the limit to prove that your calculation is correct.

Solution. First the calculation

$$\begin{aligned}
\lim_{h \to 0} \frac{f(3+h) - f(3)}{h} &= \lim_{h \to 0} \frac{\sqrt{3+h+1} - \sqrt{3+1}}{h} \\
&= \lim_{h \to 0} \frac{\sqrt{4+h} - \sqrt{4}}{h} \\
&= \lim_{h \to 0} \frac{\sqrt{4+h} - \sqrt{4}}{h} \cdot \frac{\sqrt{4+h} + \sqrt{4}}{\sqrt{4+h} + \sqrt{4}} \\
&= \lim_{h \to 0} \frac{4+h-4}{h(\sqrt{4+h} + \sqrt{4})} \\
&= \lim_{h \to 0} \frac{1}{\sqrt{4+h} + \sqrt{4}} \\
&= \frac{1}{\sqrt{4} + \sqrt{4}} \\
&= \frac{1}{4}.
\end{aligned}$$

Next the $\epsilon - \delta$ proof: Let $\epsilon > 0$. It turns out that, depending on the range of h, the algebra requires that we choose δ less than a multiple of ϵ so it will be fine to select $0 < \delta < \epsilon$ and $\delta < 1$ where the latter condition is to put a range on h. Then if $|h| < \delta$ we have:

$$\begin{aligned}
\left|\frac{f(3+h)-f(3)}{h} - \frac{1}{4}\right| &= \left|\frac{\sqrt{4+h}-\sqrt{4}}{h} - \frac{1}{4}\right| \\
&= \left|\frac{4\sqrt{4+h}-4\sqrt{4}-h}{4h}\right| \\
&= \left|\frac{4\sqrt{4+h}-(8+h)}{4h}\right| \\
&= \left|\frac{4\sqrt{4+h}-(8+h)}{4h} \cdot \frac{4\sqrt{4+h}+(8+h)}{4\sqrt{4+h}+(8+h)}\right| \\
&= \left|\frac{16\cdot(4+h)-(8+h)^2}{4h(\sqrt{4+h}+(8+h))}\right| \\
&= \left|\frac{64+16h-64-16h-h^2}{4h(4\sqrt{4+h}+(8+h))}\right| \\
&= \left|\frac{-h^2}{4h(4\sqrt{4+h}+(8+h))}\right| \\
&= \left|\frac{h}{4(4\sqrt{4+h}+(8+h))}\right| \\
&< \left|\frac{h}{4(4\sqrt{4-1}+(8-1))}\right| \\
&< \left|\frac{h}{(16\sqrt{3}+(28))}\right| \\
&< \frac{|h|}{(16\sqrt{3}+(28))} < \frac{\delta}{(16\sqrt{3}+(28))} < \epsilon.
\end{aligned}$$

Where the third step up from the bottom follows from $|h| < 1$ so $-1 < h < 1$ and the value of $h = -1$ makes the fraction into its largest possible value in that range.

□

Problem 5. For each positive integer n let

$$f_n(x) = \sqrt{x + \frac{x}{n}}.$$

Prove that the sequence $\{f_n\}_{n=1}^{\infty}$ converges uniformly over the interval $[2, 5]$.

Proof. We claim that the limit is the function $f(x) = \sqrt{x}$. To prove this let $\epsilon > 0$ and then choose $N > \frac{5}{2\sqrt{2}\epsilon}$ so that $\frac{1}{N} < \frac{2\sqrt{2}\epsilon}{5}$. Then if $n > N$ we have,

$$\begin{aligned} \left|\sqrt{x+\frac{x}{n}} - \sqrt{x}\right| &= \sqrt{x+\frac{x}{n}} - \sqrt{x} \\ &= \frac{x + \frac{x}{n} - x}{\sqrt{x+\frac{x}{n}} + \sqrt{x}} \\ &= \frac{\frac{x}{n}}{\sqrt{x+\frac{x}{n}} + \sqrt{x}}. \end{aligned}$$

We can remove the absolute value signs in the first like because $x \geq 2$.

The maximum value of the numerator is $\frac{5}{n}$ and the denominator is always bigger than $\sqrt{2} + \sqrt{2}$. So,

$$\begin{aligned} \left|\sqrt{x+\frac{x}{n}} - \sqrt{x}\right| &= \frac{\frac{x}{n}}{\sqrt{x+\frac{x}{n}} + \sqrt{x}} \\ &< \frac{1}{n}\frac{5}{2\sqrt{2}} \\ &< \frac{1}{N}\frac{5}{2\sqrt{2}} \\ &< \frac{2\sqrt{2}\epsilon}{5} \cdot \frac{5}{2\sqrt{2}} = \epsilon. \end{aligned}$$

□

Extra credit: Prove that the sequence converges uniformly over the interval $[0, b]$ for every $b > 0$.

Problem 6. Consider the function g where

$$g(x) = \begin{cases} 0 & \text{if } x < 2 \\ 2 & \text{if } 2 \leq x \leq 3 \\ 5 & \text{if } 3 < x \end{cases}$$

For each of the following functions f, determine the value of $\int_0^5 f dg$. Then use the $\epsilon - \delta$ definition of the integral to prove that your value is correct.

(a) $f(x) = 4x^2$

(b) $f(x) = \sqrt{x}$.

Proof. I'm going to argue that if f is continuous on $[0,5]$ then $\int_{[0,5]} f dg = 2f(2) + 3f(3)$.

Let $\epsilon > 0$. Since f is continuous there exists a number $\hat{\delta} > 0$ so that if $|p - x| < \delta$ then $|f(p) - f(x)| < \frac{\epsilon}{6}$. Then let $S : \{0 = x_0 < x_1 < \ldots < x_n = 5\}$ be a subdivision of $[0,5]$ with $\text{mesh}(S) < \delta$; there is some integer k so that $x_{k-1} < 2 \leq x_k$ and an integer j so that $x_{j-1} \leq 3 < x_j$. Then we have

$$\begin{aligned} & \Big| \sum_{i=1}^{n} f(x_i^*)(g(x_i) - g(x_{i-1})) - 2f(2) - 3f(3) \Big| = \\ & |f(x_k^*)(g(x_{k-1}) - g(x_k)) + f(x_j^*)(g(x_{j-1}) - g(x_j)) - 2f(2) - 3f(3)| = \\ = \; & |f(x_k^*)(2) + f(x_j^*)(3) - 2f(2) - 3f(3)| \\ = \; & |2(f(x_k^*) - f(2)) + 3(f(x_j^*) - f(3))|. \end{aligned}$$

Since $\text{mesh}(S) < \delta$ we have $|x_k^* - 2| < \delta$ and $|x_j^* - 3| < \delta$ so $|f(x_k^*) - f(2)| < \frac{\epsilon}{6}$ and $|f(x_j^*) - f(3)| < \frac{\epsilon}{6}$. So

$$\begin{aligned} |2(f(x_k^*) - f(2)) + 3(f(x_j^*) - f(3))| & \leq |2(f(x_k^*) - f(2))| + |3(f(x_j^*) - f(3))| \\ & < 2\frac{\epsilon}{6} + 3\frac{\epsilon}{6} < \epsilon. \end{aligned}$$

□

Problem 7.

(a) Prove that if f is an increasing function then the variation function satisfies:

$$\mathcal{V}_a^x f = f(x) - f(a).$$

Proof. For an arbitrary subdivision $S = \{a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b\}$ of the interval $[a,b]$, since f is increasing,

$$\begin{aligned} \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| &= |f(x_1) - f(x_0)| + |f(x_2) - f(x_1)| \\ & \quad + \ldots + |f(x_{n-1}) - f(x_{n-2})||f(x_n) - f(x_{n-1})| \\ &= f(x_1) - f(a) + f(x_2) - f(x_1) \\ & \quad + \ldots + f(x_{n-1}) - f(x_{n-2}) + f(b) - f(x_{n-1}) \\ &= f(b) - f(a). \end{aligned}$$

So, since the value of the sum does not depend on the subdivision selected we have,

$$\mathcal{V}_a^x f \quad = \quad f(x) - f(a).$$

□

(b) Given the function $f(x) = x(x-2)(x-3)$. Calculate the function $V(x) = \mathcal{V}_{-1}^x f$ and sketch its graph.

Solution. An easy calculations, setting the derivative equal to zero, gives the x-coordinate of the local maximum and minimum:

$$\begin{array}{rl} r_1 = \dfrac{5-\sqrt{7}}{3} & r_2 = \dfrac{5+\sqrt{7}}{3} \\ r_1 \approx 0.784749563 & r_2 \approx 2.54858377 \\ f(r_1) \approx 2.112611791 & f(r_2) \approx -0.631130309. \end{array}$$

The function is decreasing between r_1 and r_2 and is otherwise increasing. This means that:

$$\mathcal{V}_{-1}^x = \begin{cases} f(x) - f(-1) & \text{if } -1 \le x < r_1 \\ 2f(r_1) - f(-1) - f(x) & \text{if } r_1 \le x < r_2 \\ 2f(r_1) - 2f(r_2) - f(-1) + f(x) & \text{if } r_2 \le x. \end{cases}$$

Using the above approximations this gives us:

$$\mathcal{V}_{-1}^x = \begin{cases} f(x) + 12 & \text{if } -1 \le x < r_1 \\ 16.22522358 - f(x) & \text{if } r_1 \le x < r_2 \\ 17.4874842 + f(x) & \text{if } r_2 \le x. \end{cases}$$

□

Extra credit problem:

(a) Prove that the union of a finite number of nowhere dense sets is nowhere dense. [Hint: use induction.]

(b) Prove that if the set M has a finite number of limit points then it is nowhere dense.