AlgebraNotes03
Exercises
Creating the Rational from the Integers

As usual, lt $\mathbb{Z}$ denote the positive integers. Let Q denote the following subset $\mathbb{Z} \times \mathbb{Z}$:

$$Q = \{(x, y) | \ y \neq 0\}.$$

Define the relation $\sim$ on Q by $(a, b) \sim (c, d)$ if and only if

$$ad = bc.$$

Exercise 1. Prove that $\sim$ is an equivalence relation. [Comment: since we are creating the rationals, we cannot use the properties of the rationals to verify that $\sim$ is an equivalence relation. You may only use the axioms of the integers. So you may not use the fact that integers have multiplicative inverses; in fact this will be "proven" below. The replacement for not having a multiplicative inverse is the cancellation axiom (# B5 in the axioms provided).]

Let $\mathbb{Q}$ denote the set of equivalence classes for the relation $\sim$. As usual we will let $[(a, b)]$ denote the equivalence class that contains (a, b).

Exercise 2. Define the following operation $\oplus$ on $\mathbb{Q}$.

$$[(a, b)] \oplus [(c, d)] = [(ad + bc, bd)]$$

where $+$ and $\cdot$ denote the usual addition and multiplication of the integers $\mathbb{Z}$ respectively (see the axioms of the integers) and xy denotes $x \cdot y$.

Prove that the operation $\oplus$ is well defined.

Exercise 3. Prove that $(\mathbb{Q}, \oplus)$ is a group; prove also that this group is Abelian.

Exercise 4. Prove that $(\mathbb{Q}, \oplus)$ contains an isomorphic copy of the group $(\mathbb{Z}, +)$.

Exercise 5. Define the operation $\otimes$ on $\mathbb{Q}$ as follows:

$$[(a, b)] \otimes [(c, d)] = [(ac, bd)].$$

Prove that $\otimes$ is well defined.

Exercise 5'. Define the operation $\boxplus$ on $\mathbb{Q}$ as follows:

$$[(a,b)] \boxplus [(c,d)] \quad = \quad [(a+c, b+d)].$$

Prove that $\boxplus$ is not well defined.

Although $(\mathbb{Q}, \otimes)$ is not a group (make sure you know why). We have:

Exercise 6. Prove that $(\mathbb{Q} - \{[(0,1)]\}, \otimes)$ is an Abelian group.