AlgebraNotes03
Normal Subgroups and Homomorphisms

Definition. Suppose that $(G, \star)$ is a group and $\sim$ is an equivalence relation on G. Suppose further that we define an operation $*$ on the equivalence classes by $[x] * [y] = [x \star y]$ (where [x] denotes the equivalence class of x.) Then the operation $*$ is said to be well defined if it is true that when $x \sim a$ and $y \sim b$ we have $x \star y \sim a \star b$.

Suppose that each of G_1 and G_2 are groups with equivalence relations $\sim_1$ and $\sim_2$ respectively; suppose also that $\varphi : G_1 \to G_2$ is a function. Then the function F from the equivalence classes of $\sim_1$ to the equivalence classes of $\sim_2$ "defined" by $F([x]_1) = [\varphi(x)]_2$ is said to be well defined if it is true that when $x \sim_1 a$ we have $\varphi(x) \sim_2 \varphi(a)$.

Exercise 1. Determine if the following are well-defined functions. For n a positive integer, define the equivalence relation $\sim_n$ on the integers by $x \sim_n y$ if and only if $n|(y - x)$; we use the notation $\mathbb{Z}_n$ to denote the equivalence classes of $\sim_n$.

a.) $f : \mathbb{Z}_5 \to \mathbb{Z}_5$ with $f([x]_5) = [7x]_5$;
b.) $g : \mathbb{Z}_5 \to \mathbb{Z}_6$ with $f([x]_5) = [7x]_6$;
c.) $f : \mathbb{Z}_{10} \to \mathbb{Z}_5$ with $f([x]_{10}) = [7x]_5$.

Definition. If G is a group and H is a subgroup of G then H is said to be a normal subgroup if and only if its left coset is equal to its right coset: $gH = Hg$.

Theorem 3.1. The subgroup H of the group G is normal if and only if for each $g \in G$, $gHg^{-1} = H$.

Definition. If G_1 and G_2 are groups and $\varphi : G_1 \to G_2$ is a homomorphism then the kernel, $\text{Ker}(\varphi)$ is

$$\text{Ker}(\varphi) \quad = \quad \{x \in G_1 | \varphi(x) = e_2\}$$

where e_2 is the identity element of G_2.

Theorem 3.2. If G_1 and G_2 are groups and $\varphi : G_1 \to G_2$ is a homomorphism then the kernel, $\text{Ker}(\varphi)$ is a normal subgroup of G_1.

Theorem 3.21. Suppose that G and H are groups and $\varphi : G \to H$ is a homomorphism. Then for each $x \in G$, $\varphi(x^{-1}) = (\varphi(x))^{-1}$.

Exercises 2. Verify that the following functions are homomorphisms and find the kernels.

a.) $\varphi : (\mathbb{Z}_{12}, +_{12}) \to (\mathbb{Z}_4, +_4)$ defined by $\varphi([x]_{12}) = [x]_4$.

b.) $\varphi : (C_\infty([0,1]), +) \to (C_\infty([0,1]), +)$ defined by $\varphi(f) = f'$. (Where $C_\infty([0,1])$ denotes all the functions on the interval $[0,1]$ which are n-times differentiable for all positive integers n.)

c.) $\varphi : (C_\infty([0,1]), +) \to (\mathbb{R}, +)$ defined by $\varphi(f) = \int_0^1 f(t)dt$.

d.) $\varphi : (\mathbb{Z}_8, +_8) \to (\mathbb{Z}_5 - \{0\}, \cdot_5)$ defined by $\varphi([x]_8) = [3^x]_5$.

Exercise 3. Prove the following statements [I think that they are all true.]

a.) Suppose G is a finite group and $\varphi : G \to \hat{G}$ is a homomorphism (not necessarily onto) the group $\hat{G}$. Then

$$|\varphi(G)| \Big| |G|.$$

b.) Suppose $\hat{G}$ is a finite group and $\varphi : G \to \hat{G}$ is a homomorphism (not necessarily onto) from the group G into $\hat{G}$. Then

$$|\varphi(G)| \Big| |\hat{G}|.$$

c.) Suppose G is a finite group and $\varphi : G \to \hat{G}$ is a homomorphism from G onto the group $\hat{G}$. Then $\hat{G}$ is abelian if and only if $xyx^{-1}y^{-1} \in \mathrm{Ker}(\varphi)$ for all $x, y \in G$.

d.) If each of H and K is a normal subgroup of the group G, then $H \cap K$ is a normal subgroup of G. What about $H \cup K$?

Definition. Suppose that H is a subgroup of the group G. Then $HxHy$ means Hxy for $x, y \in G$. [Recall that if G is a group with operation $*$ then xy means $x * y$.]

Exercise 4. Let $G = (\mathbb{Z}, +)$ and $H = \{5n |\ n \in \mathbb{Z}\}$; then H is a subgroup of G.

a.) Show that the operation $\oplus$ defined by $(H+x) \oplus (H+y) = H + (x+y)$ is well defined.

b.) Write the "addition" table for $\{H + x| \ x \in \mathbb{Z}\}$ with the operation $\oplus$ and show that it's a group.

Definition Suppose that A and B are subsets of the group $(G, \star)$; define $A\hat{\star}B$ by:

$$A\hat{\star}B = \{a \star b | a \in A, b \in B\}.$$

Note that $\star$ may be, for example, multiplication or addition in the positive reals or integers respectively.

Lemma 3.3. If H is a normal subgroup of $(G, \star)$ then,

$$(Hx\hat{\star}Hy = Hxy.$$

[Recall that xy means $x \star y$, so the above may be rewritten:

$$(H \star x)\hat{\star}(H \star y) = H \star (x \star y).$$

I use $\hat{\star}$ because $\star$ and $\hat{\star}$ are two different operators, one is defined on the set G the other is defined on cosets of the subgroup H of G. In the future we will use the same symbol for both operators and in fact we will often rewrite , $(H \star x)\hat{\star}(H \star y) = H \star (x \star y)$ as $HxHy = Hxy$; especially when we are working in the general setting.]

Theorem 3.3. Suppose that H is a normal subgroup of $(G, \star)$. Then the operation $Hx \star Hy = Hxy$ is well defined. [I.e. $(H \star x)\hat{\star}(H \star y) = H \star (x \star y)$.]

Theorem 3.4. Let $(G, \cdot)$ be a group and let G/H denote the collection of cosets of H in G. Suppose that H is a normal subgroup of G. Then the operation (with respect to $\cdot$, or more precisely, with respect to $\hat{\cdot}$) defined in Theorem 3.3 is a group operation on the set G/H of cosets of H in G.

Theorem 3.5. Suppose that H is a normal subgroup of G and let $\varphi : G \to G/H$ be defined by $\varphi(x) = Hx$ then φ is a homomorphism and $\mathrm{Ker}(\varphi) = H$.

Theorem 3.6. Suppose that G and J are groups and $\varphi : G \to J$ is a homomorphism. Then $H = \{\varphi(g) \mid g \in G\}$ is a group.

Use theorems 3.2, 3.3, 3.4 and 3.6 to prove the first isomorphism theorem:

Theorem 3.7. Let G and J be groups and let $\varphi : G \to J$ be a homomorphism. Let $K = ker(\varphi)$ and $H = \{\varphi(g) \mid g \in G\}$. Then H is isomorphic to G/K.

Exercise 5. In the following assume that G is a group.

a.) The center $Z(G)$ of a group G is defined by:

$$Z(G) = \{x|xg = gx \text{ for all } g \in G\}.$$

Show that $Z(G)$ is abelian.

b.) The commutator $C(G)$ of a group G is defined by:

$$C(G) = \{xyx^{-1}y^{-1}|x, y \in G\}.$$

Show that if H is a subgroup of G containing $C(G)$ then H is a normal subgroup of G.

c.) Suppose that H is a normal subgroup of G. Then G/H is abelian if and only if $C(G) \subset H$.