AlgebraNotes04
Isomorphism Theorems.

Lemma 4.1. Suppose that N is a normal subgroup of the group G then $\gamma : G \to G/N$ defined by $\gamma(x) = Nx$ is called the canonical homeomorphism. [You should prove that it is a homomorphism.] For $L \subset G$ define $\gamma(L) = \{\gamma(x)|x \in L\}$.

1. If L is a normal subgroup of G that contains N then $\gamma(L)$ is a normal subgroup of G/N.
2. If L_1 and L_2 are normal subgroups of G containing N and $L_1 \neq L_2$, then $\gamma(L_1) \neq \gamma(L_2)$.
3. If K is a normal subgroup of G/N then there is a normal subgroup L of G containing N so that $\gamma(L) = K$.

Definition. If N and H are two subsets of the group G then $NH = \{xh|x \in N, h \in H\}$.

Lemma 4.2. Suppose that N is a normal of the group G and H is a subgroup of G. Then:

1. $NH = HN$.
2. NH is a subgroup of G.
3. If H is a normal subgroup of G then NH is also a normal subgroup of G.
4. If H is a normal subgroup of G then $N \cap H$ is also a normal subgroup of G.

use these lemmas to prove the second isomorphism theorem:

Theorem 4.1. Suppose that G is a group and each of N and H is a normal subgroup of G. Then HN and $H \cap N$ are normal groups and HN/N is isomorphic to $H/(H \cap N)$.

For the next three lemmas assume the situation for the first isomorphism theorem: Suppose G and H are groups and suppose $\varphi : G \to H$ is an onto homomorphism with kernel K. So we already know that K is normal and that the collection of cosets $\{Kx|x \in G\}$ form the group G/K. We also know that the function $\gamma : G \to G/K$ defined by $\gamma(x) = Kx$ is a homomorphism.

Lemma 4.4. If for each $x \in G$, we define $\mu(Kx) = \varphi(x)$ then μ is a well defined function.

Lemma 4.5. The function μ of Lemma 4.4 is a homomorphism.

Lemma 4.6. If $\nu : G/K \to H$ is an onto homomorphism so that

$$\nu(\gamma(x)) = \varphi(x)$$

then $\nu = \mu$.

Use these lemmas to prove the third isomorphism theorem.

Theorem 4.2. Suppose that each of H and K is a normal subgroup of the group G and that $K \subset H$. Then the group G/H is isomorphic to the group $(G/K)/(H/K)$.

Exercise 0. Consider the group $G = (\mathbb{Z}_{24}, +_{24})$ and let $\varphi : (\mathbb{Z}_{24}, +_{24}) \to (\mathbb{Z}_6, +_6)$ be defined by $\varphi([x]_{24} = [5x]_6$. Let K denote the kernel of φ.

a.) Show that G/K is isomorphic to $(\mathbb{Z}_6, +_6)$. b.) Find an isomorphism $h : G/K \to \mathbb{Z}_6$ and a homomorphism $f : G \to \mathbb{Z}_6$ so that $\varphi([x]_{24}) = h(f([x]_{24})$.

Exercise 1. Consider the group $G = (\mathbb{Z}_{24}, +_{24})$. Let

$$\begin{aligned} H &= \{0, 3, 6, 9, 12, 15, 18, 21\} \\ N &= \{0, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22\} \end{aligned}$$

a. Calculate the groups $H +_{24} N$ and $H \cap N$.

b. Calculate the group of cosets $(H + N)/N$ and $H/(H \cap N)$ and show that they are isomorphic.

c. Calculate the group of cosets $(H + N)/H$ and $N/(H \cap N)$ and show that they are isomorphic.

Exercise 2. Consider the group $G = (\mathbb{Z}_{19} - \{0\}, \cdot_{19})$.

a. Find the subgroup of G generated by [4] (i.e.: the subgroup $\{[4^n]_{19} | n \in \mathbb{Z}\}$.

b. Find the subgroup of G generated by [6] (i.e.: the subgroup $\{[6^n]_{19} | n \in \mathbb{Z}\}$.

[Hint: I used a spreadsheet to do this exercise, I think it appropriate to use a computer for these calculations. By my calculations, both of these are proper subgroups, in fact they are the same subgroup. Using MS Excel the operation (mod(x,n) gives the value of x mod n.]

c. Let H be the subgroup of (a.) and let $N = \{1, 7, 8, 11, 12, 18\}$. (Use a spreadsheets to show that N is also a subgroup.) Repeat Exercise 1 above with these groups and the operation mod 19 multiplication.

Exercise 3 [on the third isomorphism theorem]. Let $G = (\mathbb{Z}_{24}, +_{24})$ as in exercise 1 and let:

$$\begin{aligned} K &= \{0, 6, 12, 18\} \\ H &= \{0, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22\} \end{aligned}$$

a. Determine the following groups: $G/H, G/K, H/K, (G/K)/(H/K)$.

b. Show that the first and fourth are isomorphic.