AlgebraNotes06
Product Groups.

Definition. Suppose that G and H are groups with the operations $*$ and $\star$ respectively. Then the algebraic product of G and H, denoted by $G \times H$ is the set of elements $\{g, h)\ g \in G, h \in H\}$ with the operation $\cdot = * \times \star$ so that:

$$(g_1, h_1) \cdot (g_2, h_2) \quad = \quad (g_1 * g_2, h_1 \star h_2).$$

Theorem 6.1. If G and H are groups the the algebraic product $G \times H$ of G and H is a group.

Theorem 6.2. If G and H are groups then $G \times H$ is Abelian if and only if each of G and H is Abelian.

Exercises 6.1.

a.) Let $G = (\mathbb{Z}_3, +_3)$ and $H = (\mathbb{Z}_5, +_5)$. Determine if $G \times H$ is isomorphic to $(\mathbb{Z}_{15}, +_{15})$.

b.) Let $G = (\mathbb{Z}_2, +_2)$ and $H = (\mathbb{Z}_4, +_4)$. Determine if $G \times H$ is isomorphic to $(\mathbb{Z}_8, +_8)$.

Exercises 6.2. Consider the permutation group $G = S_5$. If α and β are two elements of G then the subgroup generated by α and β is the set of all elements of G in the form $\alpha^{n_1}\beta^{n_2}\alpha^{n_3}\beta^{n_4}\ldots\alpha^{n_{2k-1}}\beta^{n_{2k}}$, $n_i \in \mathbb{Z}$. [Note: you should verify that it is indeed a subgroup.]

a.) Let α and β be the two permutations $\alpha = (12)$ and $\beta = (345)$. Do α and β commute? Determine if the subgroup generated by α and β is isomorphic to some $(\mathbb{Z}_n, +_n)$.

b.) Repeat exercise 6.2 a with $\alpha = (13)$ and $\beta = (345)$.

Theorem 6.3. If G and H are groups then $N = \{(g, e_H)|g \in G, e_H$ is the identity of $H\}$ is a normal subgroup of $G \times H$. [Note: first prove it's a subgroup then prove it's normal.]

Theorem 6.3b. If G and H are groups and N is a normal subgroup of G then $\{(x, e)|x \in N\}$ is a normal subgroup of $G \times H$.

Exercises 6.3.

a.) Consider $(\mathbb{Z}_{10}, \cdot_{10})$. Let $H = \{z \in \mathbb{Z}_{10} | z \text{ has a multiplicative inverse}\}$. Then is H a group with the mod 10 multiplication operator? If yes is it cyclic?

b.) Repeat exercise 6.3 a. with $(\mathbb{Z}_{12}, \cdot_{12})$.

Exercises 6.4. Let G be a group.

a.) We've proven that if ℓ is a fixed element of G then $f_\ell : G \to G$ defined by $f_\ell(g) = \ell g$ is a one-to-one function. (The subscript is to tell us that the function is dependent upon our choice of ℓ.) So for each $x \neq y \in G$, f_x is likely to be a different function than f_y. Let $\mathcal{F}(G) = \{f_x | x \in G\}$. Show that $\mathcal{F}(G)$ with the composition operator is a group. [Composition operator: $(f_x \circ f_y)(g) = f_x(f_y(g))$.]

b.) Calculate $\mathcal{F}(G)$ for $G = (Z_7 - \{0\}, \cdot_7)$. Is it cyclic?

c.) Calculate $\mathcal{F}(G)$ for $G = (Z_{11} - \{0\}, \cdot_{11})$. Is it cyclic?