Section I. Exercises on isomorphism theorems.

Exercise 5.1. Do exercise 3 on the AlgebraNotes04.

Exercise 5.2. Define $\varphi : (\mathbb{Z}_{24}, +_{24}) \to (\mathbb{Z}_8, +_8)$ by $\varphi([x]_{24}) = [x]_8$.

a. Show that φ is well defined.

b. Show that φ is onto.

c. Find the kernel of φ.

d. Find the function μ defined in Lemma 4.4 and verify that it is well defined.

e. Show that μ is a homomorphism.

Exercise 5.3. Consider the group $G = (\mathbb{Z}_{19} - \{0\}, \cdot_{19})$. Use the multiplication chart I sent you earlier. Note that I will omit the subscript 19 and the brackets [...] so that $[8]_{19} \cdot [9]_{19} = [15]_{19}$ will be written $8 \cdot 9 = 15$.

a. Find the subgroup of G generated by 4 (i.e.: the subgroup $\{[4^n]_{19} | n \in \mathbb{Z}\}$ denote this subgroup by H.

b. Find the subgroup of G generated by 7 (i.e.: the subgroup $\{[7^n]_{19} | n \in \mathbb{Z}\}$, denote this subgroup by K.

c. Show that $K \subset H$.

d. Determine the following groups: $G/H, G/K, H/K, (G/K)/(H/K)$. [Write their multiplication charts.]

e. Argue that G/H is isomorphic to $(G/K)/(H/K)$.

f. Exercise from class 10/14/19: Is G cyclic? If yes find an isomorphism from G onto $(\mathbb{Z}_n, +_n)$ for the appropriate n. [Hint: does $[m] \to [2^m]$ work? - and why did I pick 2?]

Section II. Exercises on possible group structures.

Exercise 5.4. Suppose that G is a group and ℓ is a particular element of G.

a. Define $\varphi : G \to G$ by $\varphi(g) = \ell g$.

i. Prove that φ is one-to-one and onto.

ii. Prove that φ is an isomorphism if and only it $\ell = e$.

b. Define $\varphi : G \to G$ by $\varphi(g) = \ell g \ell^{-1}$. Prove that φ is an isomorphism.

Exercise 5.5. Suppose that G is a group, N is a normal subgroup of G, and $\varphi : G \to H$ is a homomorphism of G ***onto*** the (non-degenearate) group H with kernel K. Answer the following questions.

a. If $|G| = 35$, what are the possible values for $|K|$. [Recall that $|K|$ is the number of elements in K.

b. If $|G| = 36$ and $|K| = 6$ then what are the possible values of $|H|$.

c. If $|G| = 19$ then prove that φ is one-to-one.

Exercise 5.6. The number 4 is the smallest integer n so that there are non-isomorphic groups of size n.

a. For $n = 4$ find all the non-isomorphic groups of size n.

b. Repeat exercise (a) for $n = 5$ and $n = 6$.

Exercise 5.7. Find the smallest integer n so that there is a non-abelian group of size n.

Exercise 5.8. For each of the groups found in exercise 5.4 above find an integer n so that the group is a subset of S_n the group of permutations on n elements.