MATH5310 Final Exam preparation exercises. Fall 2019

1.) Let $\mathbb{Q}^+$ denote the positive rational numbers. Show that $(\{a+b\sqrt{5}|a,b \in \mathbb{Q}^+\}, \cdot)$ with the usual operatons on the reals, is a ring with a multiplicative identity and multiplicative inverses for non-zero elements.

2.) Suppose that $\psi : (\mathbb{Z}_{18}, +_{18}) \to (\mathbb{Z}_6, +_6)$ is defined by $\psi([x]_{18}) = [7x]_6$.
a.) Show that ψ is well defined and onto.
b.) Find the kernel K of ψ.
c.) Show that $\mathbb{Z}_{18}/K$ is isomorphic to $\mathbb{Z}_6$.

3.) Let γ and δ be the following elements of the permutation group:

$$\gamma = (35412) \qquad \delta = (314)(25).$$

a.) Calculate: (1) $\gamma \circ \delta$, (2) $\delta \circ \gamma$, (3) δ^{-1} and (4) γ^{-1}.
b.) Do γ and δ commute?
c.) Prove that γ generates a group isomorphic to $(\mathbb{Z}_n, +_n)$ for some n.
d.) Prove that δ generates a group isomorphic to $(\mathbb{Z}_n, +_n)$ for some n.
e.) Prove that $\beta = (123)(456)$ does not generate a group isomorphic to $(\mathbb{Z}_n, +_n)$ for any n.

4.) Consider the group $(\mathbb{Z}_{13} - \{0\}, \cdot_{13})$. (Use MS Excel to construct the multiplication chart.) Let H denote the subgroup generated by [4] and let K be the subgroup generated by [5].
a.) Calculate the groups HK and $H \cap K$.
b.) Calculate the groups HK/K and $H/(H \cap K)$ and show that they are isomorphic.

5.) Solve the following equations in Z_{13}, find all solutions.
a. $5x + 4 = 8$,
b. $(2x + 4)(3x - 5) = 0$,
c. $x^2 = 9$,
d. $x^2 = -1$.

5b.) Repeat (5) above for $(\mathbb{Z}_{15} - \{0\}, \cdot_{15})$.

6.) Show for $\mathbb{Z}_n$ that $[n-1]$ is its own multiplicative inverse. Then use this information to show that the function $f : \mathbb{Z}_{100} \to \mathbb{Z}_{100}$ defined by $f([x]) = [99x + 37]$ is one-to-one and onto.

Argue that n and $n-1$ are always relatively prime.

7.) Let R be the ring of positive integers. Let $I = \{7x | x \in R\}$.

a.) Show that I is a subgroup (under addition) of R.

b.) Calculate R/I

c.) Show that the operation $*$ on R/I defined by $(I + a) * (I + b) = I + ab$ is well defined.

d.) Show that $(R/I, +, *)$ is a ring.

e.) Find an integer n so that R/I is ring isomorphic to the ring $\mathbb{Z}_n$ and construct that isomorphism.

Sample Theorems.

8.) Suppose that G is a group and $H \subset G$ is a subgroup of G. Define the relation $\sim$ by $x \sim y$ if and only if $xy^{-1} \in H$.

a.) Prove that $\sim$ is an equivalence relation.

b.) Prove that the equivalence classes of $\sim$ are the right cosets of H.

c.) Suppose that $y \in G$. Prove that $\varphi : H \to Hy$ defined by $\varphi(h) = hy$ is one-to-one and onto.

9.) Suppose that G and H are groups and that $\varphi : G \to H$ is an onto homomorphism.

a.) Prove that the kernel K of φ is a normal subgroup of G.

b.) Prove that the function $\gamma : G/K \to H$ defined by $\gamma(Hx) = \varphi(x)$ is well defined and is an isomorphism.

10.) Suppose that G is a group. Prove the following:

a.) The identity of G is unique.

b.) If $x \in G$ then the inverse of x is unique.

c.) For all $x, y \in G$, prove that $(xy)^{-1} = y^{-1}x^{-1}$.

11.) Prove that if G is a group such that for each $x \in G$ we have $x^{-1} = x$, then G is abelian.

12.) Prove that if G is a group and H and K are normal subgroups of G, then if $K \subset H$, the set $\{Kh | h \in H\}$ is a normal subgroup of G/K.

13.) Prove that if $x \in \mathbb{Z}_n - \{0\}$ and x has no common divisor with n greater than 1, then x has a multiplicative inverse in $(\mathbb{Z}_n - \{0\}, \cdot_n)$.

State the theorem about Euler's φ function and show why this fact implies it.

14.) Suppose that G is a group and $H = \{x | xg = gx \text{ for all } g \in G\}$.

a.) Prove that H is a subgroup of G.

b.) Prove that H is abelian.

15.) State and prove Lagrange's theorem.

Look over the examples that verify the various isomorphism theorems. Be able to prove them in special situations (e.g. cyclic groups, special groups we've covered in class.)