MATH5310 Dr. Smith Make-Up Test 1, October 9, 2019.

Make sure to show all your work. You may not receive full credit if the accompanying work is incomplete or incorrect. If you do scratch work make sure to indicate scratch work - I will not take off points for errors in the scratch work if it is so labeled and will assume that the scratch work is not part of the final answer/proof. Do four of the following Five problems; you may do all five for extra credit.

Section I.

1. Let G be an arbitrary group. Prove the following three theorems about a group G.

a. The identity element of G is unique.

Proof. Most students proved it as follows: Suppose that in addition to e, we have that $\hat{e}$ is an identity. Then for an arbitrary x we have:

$$\begin{aligned} xe &= x \\ x\hat{e} &= x \\ xe &= x\hat{e} \\ x^{-1}xe &= x^{-1}x\hat{e} \\ ee &= e\hat{e} \\ e &= \hat{e}. \end{aligned}$$

However you should be aware of the following standard "slick" proof: Let e_1 and e_2 be identities of G. Then

$$\begin{aligned} e_1 &= e_1 \cdot e_2 && \text{because } e_2 \text{ is an identity} \\ &= e_2 && \text{because } e_1 \text{ is an identity.} \end{aligned}$$

$\square$

b. If $x \in G$ then the inverse of x is unique.

Proof. Suppose that each of x^{-1} and $\hat{x}$ is an inverse for $x \in G$. Then

$$\begin{aligned} x\hat{x} &= e \\ x^{-1}x\hat{x} &= x^{-1}e = x^{-1} \\ e\hat{x} &= x^{-1} \\ \hat{x} &= x^{-1}. \end{aligned}$$

□

c. If $x \in G$ and $y \in G$ then $(xy)^{-1} = y^{-1}x^{-1}$.

Proof.

$$\begin{aligned} (xy)^{-1}xy &= e \\ (xy)^{-1}xyy^{-1} &= ey^{-1} \\ (xy)^{-1}xe &= y^{-1} \\ (xy)^{-1}xx^{-1} &= y^{-1}x^{-1} \\ (xy)^{-1}e &= y^{-1}x^{-1} \\ (xy)^{-1} &= y^{-1}x^{-1}. \end{aligned}$$

□

Section II.

2. Suppose that N is a normal subgroup of G and H is a subgroup. Prove that $HN = NH$.

Proof. We will repeatedly use the fact that if N is normal, $n \in N$ and $g \in G$ then $gng^{-1} = \hat{n} \in N$.

First suppose that $x = nh \in NH$ is arbitrary with $n \in N$ and $h \in H$. Then

$$\begin{aligned} x &= nh \\ &= hh^{-1}nh \\ &= h\hat{n} \quad \text{for some } \hat{n} \in N \\ &\in HN \\ \therefore NH &\subset HN. \end{aligned}$$

Next suppose that $y = hn \in HN$ is arbitrary with $n \in N$ and $h \in H$. Then

$$\begin{array}{rcl} y & = & hn \\ & = & hnh^{-1}h \\ & = & \hat{n}h \quad \text{for some } \hat{n} \in N \\ & \in & NH \\ \therefore HN & \subset & NH \\ \therefore HN & = & NH. \end{array}$$

□

3. Suppose that N is a normal subgroup of G; $\psi : G \to G/N$ is defined by $\psi(g) = Ng$. Assume that if H is a subgroup of G containing N then $\psi(H)$ is a subgroup of G/N; prove that if H is normal then so is $\psi(H)$.

Proof. Let Ng be an arbitrary element of G/N and let $\varphi(x) = Nx \in \varphi(H)$ for some $x \in H$. Then:

$$\begin{array}{rcl} Ng\varphi(x)(Ng)^{-1} & = & NgNx(Ng)^{-1} \\ & = & NgNxNg^{-1} \\ & = & Ngxg^{-1} \\ & = & N\hat{x} \quad \text{for some } \hat{x} \in H \\ & \in & \varphi(H). \end{array}$$

Therefore $\varphi(H)$ is a normal subgroup of G/N.. □

Section III.

4. Let $\varphi : (\mathbb{Z}_{15}, +_{15}) \to (\mathbb{Z}_{20}, +_{20})$ be defined by $\varphi([x]_{15}) = [8x]_{20}$.

a. Show that φ is well defined.

Proof. Suppose that $x \sim_{15} y$. Then $15|(y - x)$ so there is an integer q so that:

$$\begin{array}{rcl} y - x & = & 15q \\ 8y - 8x & = & 8 \cdot 15q \\ & = & 120q = 20(6q) \\ \therefore & & 20 | (8y - 8x) \\ 8x & \sim_{20} & 8y \end{array}$$

so φ is well defined.

$\square$

b. Find the kernel K of φ.

Solution.

$$K = \{0, 5, 10\} \mod 15.$$

[Where $10 \mod 15$ denotes $[10]_{15}$ etc.] $\square$

c. Find the integer n so that $\mathbb{Z}_{15}/K$ is isomorphic to $(\mathbb{Z}_n, +_n)$. [Note: you don't have to prove that they are isomorphic.]

Solution. $n = 5$ is the integer that works. $\square$

5. Let $G = (\mathbb{Z}_{24}, +_{24})$; N and H be the following subgroups of G: $N = \{[6x]_{24} | x \in \mathbb{Z}\}$, $H = \{[4x]_{24} | x \in \mathbb{Z}\}$.

a. List the elements of $N +_{24} H$.

Solution. Note I will omit the subscript 24 and $\mod 24$ when the meaning is clear:

$$\begin{array}{rcl} N & = & \{0, 6, 12, 18\} \mod 24 \\ H & = & \{0, 4, 8, 12, 16, 20\} \mod 24 \\ N +_{24} H & = & \{0, 4, 8, 12, 16, 20\} \cup \{6, 10, 14, 18, 22, 26 = 2\} \\ & = & \{0, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22\}. \end{array}$$

$\square$

b. Find $N \cap H$.

Solution.

$$N \cap H = \{0, 12\} \mod 24$$

□

c. Write the addition chart for $(N +_{24} H)/N$.

	N	$N +_{24} [2]_{24}$	$N +_{24} [4]_{24}$
N	N	$N +_{24} [2]_{24}$	$N +_{24} [4]_{24}$
$N +_{24} [2]_{24}$	$N +_{24} [2]_{24}$	$N +_{24} [4]_{24}$	N
$N +_{24} [4]_{24}$	$N +_{24} [4]_{24}$	N	$N +_{24} [2]_{24}$

d. Write the addition chart for $H/(N \cap H)$.

	$(N \cap H)$	$(N \cap H) +_{24} [4]_{24}$	$(N \cap H) +_{24} [8]_{24}$
$(N \cap H)$	$(N \cap H)$	$(N \cap H) +_{24} [4]_{24}$	$(N \cap H) +_{24} [8]_{24}$
$(N \cap H) +_{24} [4]_{24}$	$(N \cap H) +_{24} [4]_{24}$	$(N \cap H) +_{24} [8]_{24}$	$(N \cap H)$
$(N \cap H) +_{24} [8]_{24}$	$(N \cap H) +_{24} [8]_{24}$	$(N \cap H)$	$(N \cap H) +_{24} [4]_{24}$

e. Argue that the groups from c and d are isomorphic.

Solution. Both groups are cyclic groups of order 3, so that must be isomorphic. □