MATH5310 Dr. Smith Test 1, September 27, 2019.

Make sure to show all your work. You may not receive full credit if the accompanying work is incomplete or incorrect. If you do scratch work make sure to indicate scratch work - I will not take off points for errors in the scratch work if it is so labeled and will assume that the scratch work is not part of the final answer/proof. Do five of the following six problems; you may do all six for extra credit.

1. Suppose φ is a homomorphism from the group G_1 to the group G_2.

a. Prove that if e_1 and e_2 are the identities of G_1 and G_2 respectively, then $\varphi(e_1) = e_2$.

Proof.

$$\begin{aligned} e_1 &= e_1 \cdot e_1 \\ \varphi(e_1) &= \varphi(e_1 e_1) \\ &= \varphi(e_1)\varphi(e_1) \\ \varphi(e_1)(\varphi(e_1))^{-1} &= \varphi(e_1)\varphi(e_1)(\varphi(e_1))^{-1} \\ e_2 &= \varphi(e_1)e_2 \\ &= \varphi(e_1) \end{aligned}$$

□

b. Prove that $\varphi(x^{-1}) = (\varphi(x))^{-1}$.

Proof. We start by using the results of (a.) above.

$$\begin{aligned} e_2 &= \varphi(e_1) \\ &= \varphi(x \cdot x^{-1}) \\ &= \varphi(x)\varphi(x^{-1}) \\ (\varphi(x))^{-1} &= (\varphi(x))^{-1}\varphi(x)\varphi(x^{-1}) \\ &= e_2\varphi(x^{-1}) \\ (\varphi(x))^{-1} &= \varphi(x^{-1}). \end{aligned}$$

□

c. Prove that the kernel of φ is a group. [Extra credit: prove that it is normal.]

Proof. Associativity: This is "inherited" from the operation of G_1.

Closure. Suppose $x, y \in ker(\varphi)$. Then: $\varphi(x) = e_2$ and $\varphi(y) = e_2$ so

$$\begin{array}{rcl} \varphi(xy) & = & \varphi(x)\varphi(y) \\ & = & e_2 e_2 \\ & = & e_2 \\ \text{So} \quad xy & \in & ker(\varphi). \end{array}$$

Identity. By (a.) above $\varphi(e_1) = e_2$ so $e_1 \in ker(\varphi)$.

Inverse. If $x \in ker(\varphi)$ then $\varphi(x) = e_2$ so

$$\varphi(x^{-1}) = (\varphi(x))^{-1} = e_2^{-1} = e_2$$

and so $x^{-1} \in ker(\varphi)$. $\square$

2. Consider $f : (\mathbb{Z}_{20}, +_{20}) \to (\mathbb{Z}_{12}, +_{12})$ defined by $f([x]_{20}) = [3x]_{12}$.

a. Show that f is well defined.

Proof. Suppose $x \sim_{20} y$. Then $y - x = 20q$ for some integer q. Then

$$\begin{array}{rcl} 3y - 3x & = & 3(y - x) \\ & = & 3 \cdot 20q \\ & = & 60q = 12 \cdot 5q \\ \text{so} \quad 3y & \sim_{12} & 3x. \end{array}$$

$\square$

b. Show that f is a homomorphism.

$$f([x]_{20}+[y]_{20}) = f([x+y]_{20}) = 3[x+y]_{12} = [3x+3y]_{12} = [3x]_{12}+[3y]_{12} = f([x]_{20})+f([y]_{20}).$$

c. Find the kernel of f.

$$ker(f) = \{[0]_{20}, [4]_{20}, [8]_{20}, [12]_{20}, [16]_{20}\}.$$

d. Is f an isomorphism? Indicate why or why not.

Answer. No. Since $|\mathbb{Z}_{20}| = 20$ and $|\mathbb{Z}_{12}| = 12$ the function cannot be one-to-one. Also since the kernel contains more than one element the function is not one-to-one. It's also straightforward to find two elements that map to the same element. □

3. Let $G = (\mathbb{Z}_{18}, +_{18})$. Let $H = \{[3x]_{18} \mid x \in G\}$ and consider the group $G/H = \{H + x| \; x \in G\}$.

a. Write the group operation table for G/H.

Answer: Observe that $H = \{[0]_{18}, [3]_{18}, [6]_{18}, [9]_{18}, [12]_{18}, [15]_{18}\}$; note that $H + [0]_{18} = H$.

	H	$H + [1]_{18}$	$H + [2]_{18}$
H	H	$H + [1]_{18}$	$H + [2]_{18}$
$H + [1]_{18}$	$H + [1]_{18}$	$H + [2]_{18}$	H
$H + [2]_{18}$	$H + [2]_{18}$	H	$H + [1]_{18}$

b. Prove that G/H is isomorphic to $\mathbb{Z}_n$ for some n. [I.e. find the n and show they are isomorphic.]

Solution: It follows from the table that G/H is a cyclic group of order 3 so it must be isomorphic to $\mathbb{Z}_3$. (You can tell it's cyclic by observing that the second row shows that $H + [1]$ generates the entire group.)

The table for $\mathbb{Z}_3$ written below matches that of G/H so they are isomorphic:

	$[0]_3$	$[1]_3$	$[2]_3$
$[0]_3$	$[0]_3$	$[1]_3$	$[2]_3$
$[1]_3$	$[1]_3$	$[2]_3$	$[0]_3$
$[2]_3$	$[2]_3$	$[0]_3$	$[1]_3$

4. Prove that if G and J are groups and $\varphi : G \to J$ is a homomorphism, then $H = \{\varphi(g) \mid g \in G\}$ is a subgroup of J.

Proof.
Associativity: This is 'inherited' from the operation in J.

Closure: Suppose $h_1, h_2 \in H$ then there exists $g_1, g_2 \in G$ so that $h_1 = \varphi(g_1), h_2 = \varphi(g_2)$. Then

$$\begin{aligned} h_1 h_2 &= \varphi(g_1)\varphi(g_2) \\ &= \varphi(g_i g_2) \\ &\in H. \end{aligned}$$

Identity: By problem (1 a.) $\varphi(e_G) = e_J$. So the identity is in H

Inverse: If $h \in H$ then there is a $g \in G$ so that $h = \varphi(g)$. Then

$$\begin{aligned} h^{-1} &= (\varphi(g))^{-1} \\ &= \varphi(g^{-1}) \qquad \text{by problem (1 b.)} \\ \text{so} \quad h^{-1} &\in H. \end{aligned}$$

□

5. Consider $\mathbb{Z}_n$.
a. Prove that the multiplication operator $[x] \cdot [y] = [xy]$ is well defined.

Proof. Let $\sim$ denote the equivalence relation $\sim_n$. Suppose that $x \sim a$ and $y \sim b$ then $n|(x-a)$ and $n|(y-b)$; this means that $x - a = np$ for some integer p and $y - b = nq$ for some integer g. So

$$\begin{aligned} x &= np + a \\ y &= nq + b \\ xy - ab &= (np+a)(nq+b) - ab \\ &= npq + naq + npb + ab - ab \\ &= n(pq + nq + np). \end{aligned}$$

So since n divides $xy - ab$ we have $xy \sim ab$ so the multiplication is well defined.

$\square$

b. Prove that, with respect to the multiplication operator, $[n-1]$ is it's own inverse.

Proof. Recall that any multiple of n is equivalent to zero and that $[1]$ is the multiplicative identity. Therefore

$$\begin{aligned}
[n-1][n-1] &= [n^2 - 2n + 1] \\
&= [n^2] - [2n] + 1] \\
&= [0] - [0] + [1] \\
&= [1].
\end{aligned}$$

$\square$

6. Let H be a normal subgroup of the group G; let G/H denote the set of cosets $\{Hx \mid x \in G\}$. Prove that the operation defined by $Hx \cdot Hy = Hxy$ is well defined.

Proof. Since the subgroup H is normal we have $gH = Hg$ for all $g \in G$; equivalently $gHg^{-1} = H$ for all $ginG$. So if $h \in H$ then $ghg^{-1} \in H$. Suppose now that $a \in Hx$ and $b \in Hy$; we need to argue that $ab \in Hxy$. So $a \in Hx$ and $b \in Hy$ gives us that there are $h_1, h_2 \in H$ so that $a = h_1 x$, $b = h_2 y$, therefore:

$$\begin{aligned}
ab &= h_1 x h_2 y \\
&= h_1 x h_2 x^{-1} x y \\
&= h_1 \hat{h} x y \quad \text{since } x h_2 x^{-1} = \hat{h} \text{ for some } \hat{h} \in H \\
&\in Hxy.
\end{aligned}$$

$\square$