Topology Math 5500/6500 Fall 2021
Dr. Smith
Project 02 Part 1

The project is due Monday Oct. 25 before class. As usual - please send me your work as a pdf file with the file name beginning with your last name.

As usual - please send me your work **on time** as a **pdf file** with the file name **beginning with your last name**; failure to do so may cause you to incur a penalty.

A short course in analysis.

We have developed the tools of topology to the point that some of the standard theorems of analysis can be proven using the theorems that we have worked on. It turns out that these theorems will hold for any ordered topological which is connected and for which intervals $[a, b]$ are compact and for which the least upper bound principle holds. For the solutions of the following problems you may use only the theorems from class and be careful not to assume that your space is metric unless the problems specifically references the real numbers (as in problem 1).

Make sure to reference the theorems that you are using in your proofs/solutions that you write up. Some of the conclusions follow almost trivially from our theorems - make sure to reference them (by number is fine) when you use them.

You may do 6 of the following 7 problems; if you do all seven I will base your grade on the best 6 out of 7.

Assume that the reals $\mathbb{R}$ has the standard topology. Assume for the following (except for problem 7) that the interval $[0, 1]$ is compact and connected.

Problem 1. We've shown that the unit interval $[0, 1]$ is compact; use Theorem 5.3 to argue that an arbitrary interval in $\mathbb{R}$ is compact.

Problem 2. Show that if $\mathbb{R}$ is connected then so is every interval.

Problem 3. Show that a subset of $\mathbb{R}$ is compact if and only if it is closed and bounded.

Problem 4. Show that every infinite and bounded set has a limit point.

Problem 5. [The intermediate value property.] Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous. Let $a < b$ and suppose that y is a number between $f(a)$ and $f(b)$. Then there is a number x between a and b so that $f(x) = y$. [Hint: use the results of problem 2.]

Problem 6. [The high point theorem.] Suppose $[a, b]$ is an interval and $f : [a, b] \to \mathbb{R}$ is continuous. Then there is a number $c \in [a, b]$ so that $f(c) \geq f(x)$ for all $x \in [a, b]$. [The value $f(c)$ is the maximum value of f over the interval $[a, b]$ and the point $(c, f(c))$ is a high point.]

Problem 7. Prove the lemma stated in class that I used to prove that the real numbers is connected. Then complete the proof that $\mathbb{R}$ is connected.

Extra Credit: Show that one can deduce the least upper bound property from that fact that $\mathbb{R}$ is connected.