**Topology Math 5500/6500 Fall 2021**
**Dr. Smith**
**Project 02 Part 1 Key**

The project is due Monday Oct. 25 before class. As usual - please send me your work as a pdf file with the file name beginning with your last name.

As usual - please send me your work **on time** as a **pdf file** with the file name **beginning with your last name**; failure to do so may cause you to incur a penalty.

**A short course in analysis.**

We have developed the tools of topology to the point that some of the standard theorems of analysis can be proven using the theorems that we have worked on. It turns out that these theorems will hold for any ordered topological which is connected and for which intervals $[a, b]$ are compact and for which the least upper bound principle holds. For the solutions of the following problems you may use only the theorems from class and be careful not to assume that your space is metric unless the problems specifically references the real numbers (as in problem 1).

**Make sure to reference the theorems that you are using** in your proofs/solutions that you write up. Some of the conclusions follow almost trivially from our theorems - make sure to reference them (by number is fine) when you use them.

You may do 6 of the following 7 problems; if you do all seven I will base your grade on the best 6 out of 7.

Assume that the reals $\mathbb{R}$ has the standard topology. Assume for the following (except for problem 7) that the interval $[0, 1]$ is compact and connected.

Problem 1. We've shown that the unit interval $[0, 1]$ is compact; use Theorem 5.3 to argue that an arbitrary interval in $\mathbb{R}$ is compact.

*Proof.* Let $[a, b]$ be an arbitrary interval in $\mathbb{R}$. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) \quad = \quad = a + (b - a)x.$$

Then $f([0,1]) = [a,b]$. For $\epsilon > 0$ if we let $\delta = \epsilon/(b-a)$ then for $|x-t| < \delta$ we have $|f(x) - f(t)| < \epsilon$. So $f$ is continuous and then the compactness of $[a,b]$ follows from Theorem 5.3. □

Problem 2. Show that if $\mathbb{R}$ is connected then so is every interval.

*Proof.* Suppose some interval $[a,b]$ is not connected and $[a,b] = H \cup K$ where $H$ and $K$ are nonempty mutually separated. (Abbreviating "without loss of generality" with "wlog.") Wlog assume $a \in H$.

Case 1. $b \in K$. Then let

$$\begin{aligned}\hat{H} &= H \cup (-\infty, a) \\ \hat{K} &= K \cup (b, \infty).\end{aligned}$$

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$$\begin{aligned}\hat{H} &= H \cup (-\infty, a) \cup (b, \infty) \\ \hat{K} &= K.\end{aligned}$$

Then it is straightforward to prove that $\hat{H}$ and $\hat{K}$ are nonempty mutually separated sets that union to $\mathbb{R}$. This contradicts the assumed connectedness of $\mathbb{R}$.

□

Problem 3. Show that a subset of $\mathbb{R}$ is compact if and only if it is closed and bounded.

*Proof.* Part 1. Suppose that the set $M$ is closed and bounded. Then there is an interval $[a,b]$ so that $M \subset [a,b]$. Since $M$ is closed and by problem 1, $[a,b]$ is compact, then by Theorem 5.1 $M$ is compact.

Part 2. Suppose that the set $M$ is compact. Then by Theorem 5.2 it is closed. Next let $G = \{(-n,n) | n \in \mathbb{N}\}$. Observe that $G$ is a covering of $\mathbb{R}$ with open sets. So $G$ covers $M$ and so by compactness some finite subcollection $G'$ of $G$ covers $M$. So if we let $N = \max\{n | (-n,n) \in G'\}$ it follows that $M \subset (-N, N)$ and hence $M$ is bounded.

□

Problem 4. Show that every infinite and bounded set has a limit point.

*Proof.* Let $M$ be an infinite and bounded set. Since it is bounded there is an interval $[a, b]$ so that $M \subset [a, b]$. Since $[a, b]$ is compact and $M$ is infinite, by Theorem 5.11, $M$ has a limit point. □

Problem 5. [The intermediate value property.] Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous. Let $a < b$ and suppose that $y$ is a number between $f(a)$ and $f(b)$. Then there is a number $x$ between $a$ and $b$ so that $f(x) = y$. [Hint: use the results of problem 2.]

*Proof.* Assume to the contrary that there is a point $y$ between $f(a)$ and $f(b)$ so that no point of $[a, b]$ maps onto it. We prove the case where $f(a) < f(b)$, the other case is similar. Let $U = \{t|y > t\}$ and let $V = \{t|y < t\}$. Observe that $U$ and $V$ are disjoint open sets and that every point of $[a, b]$ is mapped onto a point of $U$ or a point of $V$. Since $f$ is continuous $f^{-1}(U)$ and $f^{-1}(V)$ are open and by construction they are disjoint. Furthermore $a \in f^{-1}(U)$ and $b \in f^{-1}(V)$. So if we let $H = f^{-1}(U) \cap [a, b]$ and $K = f^{-1}(V) \cap [a, b]$, then $H$ and $K$ are disjoint nonempty sets; since they are subsets of disjoint open sets, they are mutually separated. Which contradicts problem 2. □

Problem 6. [The high point theorem.] Suppose $[a, b]$ is an interval and $f : [a, b] \to \mathbb{R}$ is continuous. Then there is a number $c \in [a, b]$ so that $f(c) \geq f(x)$ for all $x \in [a, b]$. [The value $f(c)$ is the maximum value of $f$ over the interval $[a, b]$ and the point $(c, f(c))$ is a high point.]

*Proof.* Suppose the hypothesis of the theorem. By problem 1, $f[a, b]$ is compact and by problem 3 it is closed and bounded. Since $f[a, b]$ is bounded it has a least upper bound $\ell$. Since it is closed the least upper bound is an element of $f[a, b]$ so there is a point $c$ so that $\ell = f(c)$ therefore since $f(x) \leq \ell$ for all $x \in [a, b]$ it follows that $f(x) \leq f(c)$ for all $x \in [a, b]$.

[Comment. Notice that all that was needed for the proof was that the domain was compact. The same proof works for $f : X \to \mathbb{R}$ for $X$ compact.] □

Problem 7. Prove the lemma stated in class that I used to prove that the real numbers is connected. Then complete the proof that $\mathbb{R}$ is connected.

*Proof.* We assumed that $[-n, n]$ was not connected and that $H$ and $K$ are mutually separated sets so that $[-n, n] = H \cup K$. We assumed that $a \in H$ and we let $M = \{x | [-n, x) \subset H\}$. Observe that if $x \in M$ and $a < t < x$, then $t \in M$. If $n \in H$ then there is an open set containing $n$ and no point of $K$ but that open set must contain a point of $M$ from which it follows from our observation that $M = [a, b) \subset H$ so $b \in H$ and so $K$ is empty. Therefore $n \notin M$ and $M$ has a least upper bound $\ell$. Let $(u, v)$ be a segment containing $\ell$; by construction $(u, \ell)$ contains a point of $M$ so $\ell$ is a limit point of $M$ and hence of $H$, so $\ell$ is not in $K$ and is in $H$, so by the least upper bound property, there is a point $t$ in $(\ell, v)$ not in $M$ so there is a point $q$ in $(\ell, t)$ not in $H$ and so $q \in K$. So $\ell$ is a limit point of $K$ but this contradicts the fact that $H$ and $K$ are supposed to be mutually separated.

Therefore for each integer $n$ the set $[-n, n]$ is connected and each of these intervals contain the point 0. So by Theorem 6.4, $\mathbb{R}$ is connected. □

Extra Credit: Show that one can deduce the least upper bound property from that fact that $\mathbb{R}$ is connected.

*Proof.* Suppose that $M$ is a bounded set and that $u$ is an upper bound to $M$. Let $L = \{u | m \leq u \text{ for all } m \in M\}$. Let $K = \mathbb{R} - L$. Observe that:

1. If $u \in L$ and $u' > u$ then $u' \in L$.
2. If $t \in K$ and $t' < t$ then $t' \in K$.
3. If $u \in L$ and $t \in K$ then $t < u$.
4. If $m \in M$ and $t < m$ then $t \in K$.
5. $L \cup K = \mathbb{R}$ and $L \cap K = \emptyset$.

Then, since $\mathbb{R}$ is connected one of the sets $K$ or $L$ contains a limit point of the other. Call this point $\ell$.

Claim 1: $\ell \in L$. For if not, then by definition of $L$ there must be a point $m \in M$ so that $\ell < m$. But then, by observations 2 and 4 above, $(-\infty, m)$ is an open set containing $\ell$ and no points of $L$ and this contradicts $\ell$ being in one of the sets and a limit point of the other.

Claim 2: no point of $L$ precedes $\ell$. For if a point $u \in L$ precedes $\ell$ then, by observations 1 and 3 above, $(u, \infty)$ is an open set containing $\ell$ and no points of $K$; so $\ell$ lies in $L$ and is not a limit point of $K$. Again this contradicts $\ell$ being in one of the sets and a limit point of the other.

Then $\ell$ is an upper bound of $M$ by Claim 1. And by Claim 2 no other upper bound of $M$ precedes $\ell$; so $\ell$ is the least upper bound of $M$.

□