**Topology Math 5500/6500 Fall 2021**
**Dr. Smith**
**Project 03 Part 1**

Problems 1 - 4 of the project is due Monday Nov. 8 before class; the rest will be due at a later date - but look them over in case you have questions about them that I can address on Monday. As usual - please send me your work as a pdf file with the file name beginning with your last name.

As usual - please send me your work **on time** as a **pdf file** with the file name **beginning with your last name**; failure to do so may cause you to incur a penalty.

## The order topology.

In the following problems assume that the topology is the order topology defined in the 08 set of notes based on what ever order is defined. By a "basic" open set, I mean an element of the basis as described in the 08 notes. Be aware (as will be shown) that the order topology is not necessarily the same as the subspace topology (see theorem 2.3).

Problem 01. Suppose $X = [0,1] \cup [2,3]$ with the order topology based on the usual order from the reals.

a.) Find two disjoint basic open sets $U$ and $V$ so that $0 \in U$ and $1 \in V$,

b.) Find two disjoint basic open sets $U$ and $V$ so that $1 \in U$ and $2 \in V$.

c.) Use the theorems we've proven in the course to argue that $X$ as a space satisfies the least upper bound axiom.

Problem 02. Suppose $X = \{x \in [0,1] | x \text{ is rational } \}$ with the usual the order topology based on the order from the reals.

a.) Find two disjoint basic open sets $U$ and $V$ so that $0 \in U$ and $1 \in V$,

b.) Show that $X$ is not connected.

c.) Argue that $X$ as a space does not satisfy the least upper bound axiom.

Problem 03. Suppose $X = \{x \in [0,1] | x \text{ is not rational } \} \cup [2,3]$ with the order topology based on the usual order from the reals.

a.) Find two disjoint basic open sets $U$ and $V$ so that $\frac{\sqrt{2}}{2} \in U$ and $2 \in V$,

b.) Argue that $X$ as a space does not satisfy the least upper bound axiom.

c.) Show that 2 is the least upper bound of the set of points of $X$ that precede it. [Note/hint: note that the order topology is not the same as the subspace topology defined earlier in the semester; see theorem 2.3.]

Problem 04. Prove Theorem 8.1.

Problem 05. Let $a < b$. Show that there is a homeomorphism (one-to-one, onto, continuous and whose inverse is also continuous) from $[0, 1]$ onto $[a, b]$. [Hint: look at my solution to problem 1 of project 02.]

## Project 03 Part 2

Due Monday Nov. 15.

### The order topology continued.

Problem 06. Suppose $X = [0, 1] \cup (2, 3]$ with the order topology based on the order $<$. (The usual order from the reals.)

a.) Argue that $X$ (with the order topology) is homeomorphic to $[0, 1]$.

b.) Show that $X$ is connected.

[You may do these in any order; depending on which one you do first, the other one may follow from it.]

### The well-ordering principle.

In the following we will be considering subsets of the real numbers. Let the symbol $<$ to denote the usual ordering of the reals. You may assume Theorem 8.2 if you need it.

Problem 07. Let $\mathbb{Z}$ denote the set of all integers (positive, negative and zero).

a.) Show that $\mathbb{Z}$ with the usual order from the reals, is not a well-ordered set.

b.) Define a different order $\prec$ on $\mathbb{Z}$ as follows (note that we define a new order $\prec$ in terms of a previously known order $<$):

Let $a$ and $b$ be elements of $\mathbb{Z}$. Then

If $0 \leq a$ and $0 \leq b$ then $a \prec b$ if and only if $a < b$. [i.e. for non-negative numbers the ordering stays the same.]

If $0 \leq a$ and $b < 0$ then $a \prec b$. [i.e. the non-negatives are less than the negatives in the new order.]

If $a < 0$ and $b < 0$ then $a \prec b$ if and only if $b < a$. [i.e. the new ordering on the negatives is the reverse of the old ordering.]
List the elements $-3, -2, -1, 0, 1, 2, 3$ in the order induced by $\prec$.

c.) Show that $\prec$ is a well-ordering on $\mathbb{Z}$.

d.) With respect to the ordering $\prec$ find the least element of the following sets:

$M_1 = X$.
$M_2 = \{x \in \mathbb{Z} | x < 0\}$.

e.) Show that if $\mathbb{Z}$ is given the order topology induced by $\prec$ then $-1$ is a limit point of $\mathbb{Z}$.

f.) Show that if $\mathbb{Z}$ is given the order topology induced by $\prec$ then $-2$ is not a limit point of $\mathbb{Z}$.

Problem 08. Let $X$ denote the following set of set real numbers and assume that $X$ has the topology induced by the usual ordering $<$:

$$X = \left\{\frac{n}{n+1}\middle| n \in \mathbb{N}\right\} \cup \left\{1 + \frac{n}{n+1}\middle| n \in \mathbb{N}\right\} \cup \left\{2 + \frac{n}{n+1}\middle| n \in \mathbb{N}\right\}.$$

a.) Show that $X$ is well ordered. [Hint: let $M \subset X$ and look at three cases depending on if $M$ does or doesn't contain points in each of the three "pieces" defining $X$.]

b.) Argue that the topology induced by the well-ordering is not the same as the subspace topology (with $X$ as a subspace of the reals with the usual topology on the reals.)

c.) Argue that $X$ has some limit points with respect to the order topology.

Problem 09.

Prove any of the theorems 8.5 through 8.10