Math 5500/6500, Fall 2021, Dr. Smith
Test 01

Instructions: The test is due by before class Monday September 27. The test is open book and open notes; this includes my notes on the website and class notes on the canvas website. You may not receive any other outside assistance and may not discuss the test with anyone. Please affirm at the beginning of your hand-in work that you have abided by these conditions.

Email to me as an attachment your solutions to the problems as a pdf file with the file name beginning with your last name: e.g. smithxyztest01.pdf.

Problem instructions: Math5500 students may omit one question (if you do all, I'll omit the one with least credit); Math6500 should do all the questions. You may assume any of the theorems in the notes or any lemmas or exercises proven in class. You must prove any other lemmas that you use. If $(X, \mathcal{T}_d)$ is a metric space then $\mathcal{T}_d$ denotes the topology and d is the metric that generates the topology. The symbol $\mathbb{R}$ denotes the real numbers with the standard topology unless otherwise indicated (as in problem part B of problem 4); $\mathbb{Q}$ denotes the rational numbers and $\mathbb{N}$ denotes the positive integers $1, 2, 3, \ldots$. If M is a set $\sup(M)$ denotes the least upper bound of M.

Problem 1. Suppose $(X, \mathcal{T}_d)$ is a metric space.

Part A. Suppose $p \in X$ and $\epsilon > 0$. Prove that if $d(x,p) < \epsilon$ and $0 < \delta < \epsilon - d(x,p)$ then:

$$B_\delta(x) \quad \subset \quad B_\epsilon(p).$$

Part B. Prove that for each $p \in X$ that the following collection $\mathcal{B}_p$ is a local basis at p:

$$\mathcal{B}_p \quad = \quad \{B_{\frac{1}{n}}(p) |\ n \in \mathbb{N}\}.$$

Problem 2. Consider the topological space $(\mathbb{R}, \mathcal{T}_d)$ with the usual metric d. Determine the sequential limit of the following sequence and prove that it is the sequential limit:

$$\left\{2 + \frac{3}{4\sqrt{2n-5}}\right\}_{n=1}^{\infty}.$$

Problem 3. Suppose $(X, \mathcal{T})$ is a Hausdorff space, $M \subset X$ and M' is the set of limit points of M. Prove that if p is a boundary point of M' then p is a limit point of M.

Problem 4. Consider the following subsets M_i of $\mathbb{R}$:

$$\begin{aligned} M_1 &= \left\{1 - \frac{1}{n}\right\}_{n=1}^{\infty} \bigcup \left\{2 + \frac{2}{n}\right\}_{n=1}^{\infty} \bigcup \left\{3 - \frac{4}{n}\right\}_{n=1}^{\infty} \\ M_2 &= \{x | x^2 < 5\} \cup [5, 7] \\ M_3 &= \{x | x \leq -7\} \cup ((2, 3) \cap \mathbb{Q}) \cup (5, 6). \end{aligned}$$

Part A. Using the standard topology of $\mathbb{R}$, determine
(i) the set of limit points of M_i;
(ii) the interior of M_i;
(iii) the boundary of M_i.

Part B. Repeat part A but with the topology $\widehat{\mathcal{T}}$ on $\mathbb{R}$ generated by the basis $\widehat{\mathcal{B}} = \{[a, b) |\ a < b\}$.

Problem 5. Suppose $(X, \mathcal{T}_d)$ is a metric space, $\{x_i\}_{i=1}^{\infty}$ is a sequence of point that has sequential limit p and $f : X \to \mathbb{R}$ is a continuous function. Prove that $f(p)$ is the sequential limit of the sequence $\{f(x_i)\}_{i=1}^{\infty}$.

Problem 6. Suppose $(X, \mathcal{T}_d)$ is a metric space, $p \in X$ and $\epsilon > 0$.

Part A. Prove that if x is a boundary point of $B_\epsilon(p)$ then $d(x, p) = \epsilon$.

Part B. Prove that if $\delta < \epsilon$ then $\overline{B_\delta(p)} \subset B_\epsilon(p)$.

Problem 7. Let $X = C([0, 1])$ denote the set of continuous functions from $[0, 1]$ into itself. For each $n \in N$ define f_n as follows:

$$f_n(x) = \begin{cases} 1 - nx & \text{if } 0 \leq x \leq \frac{1}{n} \\ 0 & \text{if } \frac{1}{n} < x \leq 1 \end{cases}$$

and let h be the identically 0 function: $h(x) = 0$ for all $x \in [0, 1]$.

Part A: Define a metric d on X by $d(f, g) = \int_0^1 |f - g| dx$. Show that with this metric the point h is a sequential limit point of the sequence $\{f_n\}_{n=1}^{\infty}$.

Part B: Define a metric ρ on X by $\rho(f, g) = \sup\{|f(x) - g(x)| \mid x \in [0, 1]\}$. Show that with this metric the point h is a not a sequential limit point of the sequence $\{f_n\}_{n=1}^{\infty}$.