Math 5500/6500, Fall 2021, Dr. Smith
Test 01, Key

Instructions: The test is due by before class Monday September 27. The test is open book and open notes; this includes my notes on the website and class notes on the canvas website. You may not receive any other outside assistance and may not discuss the test with anyone. Please affirm at the beginning of your hand-in work that you have abided by these conditions.

Email to me as an attachment your solutions to the problems as a pdf file with the file name beginning with your last name: e.g. smithxyztest01.pdf.

Problem instructions: Math5500 students may omit one question (if you do all, I'll omit the one with least credit); Math6500 should do all the questions. You may assume any of the theorems in the notes or any lemmas or exercises proven in class. You must prove any other lemmas that you use. If $(X, \mathcal{T}_d)$ is a metric space then $\mathcal{T}_d$ denotes the topology and d is the metric that generates the topology. The symbol $\mathbb{R}$ denotes the real numbers with the standard topology unless otherwise indicated (as in problem part B of problem 4); $\mathbb{Q}$ denotes the rational numbers and $\mathbb{N}$ denotes the positive integers $1, 2, 3, \ldots$. If M is a set $\sup(M)$ denotes the least upper bound of M.

Problem 1. Suppose $(X, \mathcal{T}_d)$ is a metric space.

Part A. Suppose $p \in X$ and $\epsilon > 0$. Prove that if $d(x,p) < \epsilon$ and $0 < \delta < \epsilon - d(x,p)$ then:

$$B_\delta(x) \quad \subset \quad B_\epsilon(p).$$

Proof. Assume the hypothesis and let $t \in B_\delta$ then

$$\begin{aligned} d(t,x) &< \delta \\ &< \epsilon - d(x,p) \\ d(t,x) + d(x,p) &< \epsilon \\ d(t,p) &< d(t,x) + d(x,p) < \epsilon \\ \therefore \quad & t \in B_\epsilon(p) \\ \therefore \quad & B_\delta(x) \subset B_\epsilon(p). \end{aligned}$$

□

Part B. Prove that for each $p \in X$ that the following collection $\mathcal{B}_p$ is a local basis at p:

$$\mathcal{B}_p = \{B_{\frac{1}{n}}(p) |\ n \in \mathbb{N}\}.$$

Comment: Some students used the fact that $\{B_\epsilon(p)\}_{\epsilon>0}$ is a local basis; this is fine. The following essentially reproves this, which is what many of you did; and that's fine too.

Proof. Let $p \in X$ and let U be an open set containing p. Then there is $q \in X$ and $\epsilon > 0$ that determine a basis element $B_\epsilon(q)$ that contains p and is a subset of U. Let n be such that $\frac{1}{n} < \epsilon - d(p, q)$. Then by part A of this problem we have

$$\begin{aligned} B_{\frac{1}{n}}(p) &\subset B_\epsilon(q) \\ &\subset U. \end{aligned}$$

□

Problem 2. Consider the topological space $(\mathbb{R}, \mathcal{T}_d)$ with the usual metric d. Determine the sequential limit of the following sequence and prove that it is the sequential limit:

$$\left\{2 + \frac{3}{4\sqrt{2n-5}}\right\}_{n=1}^{\infty}.$$

Solution. The limit is 2. For $\epsilon > 0$, select any integer N_ϵ so that

$$N_\epsilon > \frac{1}{2}\left(\frac{3}{4\epsilon}\right)^2 + \frac{5}{2}.$$

(In fact, any integer greater than this, such as $\frac{1}{\epsilon^2} + \frac{5}{2}$, will also work.)

Then prove that for every $n > N_\epsilon$ we have

$$\left|2 + \frac{3}{4\sqrt{2n-5}} - 2\right| < \epsilon.$$

□

Problem 3. Suppose $(X, \mathcal{T})$ is a Hausdorff space, $M \subset X$ and M' is the set of limit points of M. Prove that if p is a boundary point of M' then p is a limit point of M.

Proof. Some students found different proofs using some of the theorems proven in class.

Let p is a boundary point of M' and let U be an arbitrary open set containing p. Then U contains a point of M' and a point not in M'. Since it contains a point x in M' it also contains a point t of M (which may or may not be p) distinct from x. If $t \neq p$ then we have found a point of M in U distinct from p. If $t = p$ then by the Hausdorff condition, there are disjoint open sets V and W containing t and x and respectively. Then $W \cap U$ is an open set and so contains a point distinct from x in M. This point is also in U and cannot be p and so, in either case, U contains a point of M distinct from p and therefore p is a limit point of M. $\square$

Problem 4. Consider the following subsets M_i of $\mathbb{R}$:

$$\begin{aligned} M_1 &= \left\{1 - \frac{1}{n}\right\}_{n=1}^{\infty} \bigcup \left\{2 + \frac{2}{n}\right\}_{n=1}^{\infty} \bigcup \left\{3 - \frac{4}{n}\right\}_{n=1}^{\infty} \\ M_2 &= \{x | x^2 < 5\} \cup [5, 7] \\ M_3 &= \{x | x \leq -7\} \cup ((2, 3) \cap \mathbb{Q}) \cup (5, 6). \end{aligned}$$

Part A. Using the standard topology of $\mathbb{R}$, determine
(i) the set of limit points of M_i;
(ii) the interior of M_i;
(iii) the boundary of M_i.

Part B. Repeat part A but with the topology $\widehat{\mathcal{T}}$ on $\mathbb{R}$ generated by the basis $\widehat{\mathcal{B}} = \{[a, b) |\ a < b\}$.

Problem 5. Suppose $(X, \mathcal{T}_d)$ is a metric space, $\{x_i\}_{i=1}^{\infty}$ is a sequence of point that has sequential limit p and $f : X \to \mathbb{R}$ is a continuous function. Prove that $f(p)$ is the sequential limit of the sequence $\{f(x_i)\}_{i=1}^{\infty}$.

Proof. Assume the hypothesis of the theorem and suppose that U is an open set in Y containing $f(p)$. Then, by the definition of continuity, $f^{-1}(U)$ is an open set in X and it contains p. So there exists an integer N so that if $n > N$, then $x_n \in f^{-1}(U)$. If $x_n \in f^{-1}(U)$ then $f(x_n) \in U$. Therefore we have found an integer N so that for $n > N$, we have $f(x_n) \in U$. Therefore, $f(p)$ is the sequential limit of the sequence $\{f(x_i)\}_{i=1}^{\infty}$. $\square$

Problem 6. Suppose $(X, \mathcal{T}_d)$ is a metric space, $p \in X$ and $\epsilon > 0$.

Part A. Prove that if x is a boundary point of $B_\epsilon(p)$ then $d(x,p) = \epsilon$.

Proof Outline. Case 1. Assume $d(x,p) < \epsilon$; then for $\delta > 0$ so that $\delta < \epsilon - d(x,p)$, by part A of problem 1 we have $B_\delta(x) \subset B_\epsilon(p)$ so x cannot be a boundary point.

Case 2. Assume $d(x,p) > \epsilon$ use an argument similar to the one for problem 1 A to show that if δ is a positive number so that $\delta < d(x,p) - \epsilon$ then $B_\delta(x)$ does not intersect $B_\epsilon(p)$ and so is not a boundary point of $B_\epsilon(p)$.

The only possibility left is that $d(x,p) = \epsilon$. □

Part B. Prove that if $\delta < \epsilon$ then $\overline{B_\delta(p)} \subset B_\epsilon(p)$.

Proof Outline. Argue the following lemma: $\overline{M} = M \cup \mathrm{Bd}(M)$. Then part B follow from part A plus this lemma. □

Problem 7. Let $X = C([0,1])$ denote the set of continuous functions from $[0,1]$ into itself. For each $n \in N$ define f_n as follows:

$$f_n(x) = \begin{cases} 1 - nx & \text{if } 0 \le x \le \frac{1}{n} \\ 0 & \text{if } \frac{1}{n} < x \le 1 \end{cases}$$

and let h be the identically 0 function: $h(x) = 0$ for all $x \in [0,1]$.

Part A: Define a metric d on X by $d(f,g) = \int_0^1 |f - g| dx$. Show that with this metric the point h is a sequential limit point of the sequence $\{f_n\}_{n=1}^{\infty}$.

Proof Outline. Show that $d(h, f_n) = \frac{1}{2n}$ then for $N_\epsilon > \frac{1}{2\epsilon}$ we have: if $n > N_\epsilon$ then $d(h, f_n) < \epsilon$.

□

Part B: Define a metric ρ on X by $\rho(f,g) = \sup\{|f(x) - g(x)| \mid x \in [0,1]\}$. Show that with this metric the point h is a not a sequential limit point of the sequence $\{f_n\}_{n=1}^{\infty}$.

Proof Outline. Argue that $\rho(h, f_n) = 1$. Then for $\epsilon = \frac{1}{2}$, $B_\epsilon(h)$ (with respect to the ρ metric) contains h but no points of the sequence $\{f_n\}_{n=1}^{\infty}$. □