**Topology 5500/6500 Fa21 - Dr. Smith**
**Section 01 - Various Proofs**

Theorem 1.1. Let $(X, \mathcal{T}(X))$ be a topological space. Let $U \in \mathcal{T}(X)$ be an open set, then $U$ does not contain any of its boundary points.

*Proof.* Let $U$ be an open set and suppose $p \in \mathrm{Bd}(U)$. If $O$ is an open set containing the boundary point $p$, then $O$ contains a point of $U$ and a point not in $U$. But $U$ is an open set so it cannot contain $p$, since in order for it to do so, it would have to contain points not in $U$ which is a contradiction. □

Theorem 1.2. Let $(X, \mathcal{T}(X))$ be a topological space. If $M \subset X$ then $X - Bd(M)$ is open.

*Proof.* Suppose the hypothesis of the theorem and suppose that $p \in X - Bd(M)$.

Case 1: $p \in M$. Since $p$ is not a boundary point of $M$. There must exist an open set $U_p$ so that either $U_p$ contains no point of $M$ or no point of $X - M$. Since $p \in M$ it follows that $p \in U_p \subset M$ and so $U_p$ does not contain any boundary points of $M$. So $U_p \subset X - Bd(M)$.

Case 2: $p \notin M$. Then $p \in X - M$. Since $p$ is not a boundary point of $M$. There must exist an open set $U_p$ so that either $U_p$ contains no point of $M$ or no point of $X - M$. Since $p \in X - M$ it follows that $p \in U_p \subset X - M$ and so, as in case 1, $U_p$ does not contain any boundary points of $M$ and again we have $U_p \subset X - Bd(M)$.

Since $(X, \mathcal{T}(X))$ be a topological space, it follows that the set

$$O = \bigcup_{p \in X - \mathrm{Bd}(M)} U_p$$

is open. Since each point of $X - \mathrm{Bd}(M)$ is in $O$ and each $U_p$ is a subset of $X - \mathrm{Bd}(M)$ it follows that $O = X - \mathrm{Bd}(M)$. □

Exercise. Let $(X, \mathcal{T}(X))$ be a topological space and $M \subset X$. Then $\text{Bd}(M) = \text{Bd}(X - M)$.

*Proof.* Suppose the hypothesis of the theorem so that $M \subset X$. Let $p \in \text{Bd}(M)$. Then if $U$ is an open set containing $p$, $U$ contains a point of $M$ and a point of $X - M$. Therefore

$$\text{Bd}(M) \quad \subset \quad \text{Bd}(X - M). \tag{1}$$

Let $p \in \text{Bd}(X - M)$. Then if $U$ is an open set containing $p$, $U$ contains a point of $X - M$ and a point of $M$. Therefore

$$\text{Bd}(X - M) \quad \subset \quad \text{Bd}(M). \tag{2}$$

Together equations (1) and (2) imply that $\text{Bd}(M) = \text{Bd}(X - M)$.

□