MATH5500/6500 Final Exam
Dr. Smith.

MATH5500 The test is due at the end of our scheduled final exam period, Thursday December 9, 2:30 pm. The test is open notes; you may use your notes, any of the class notes saved on the canvas site and any of the notes on the class web site. You may not use any other resource nor receive any other outside assistance; you may not discuss the test with anyone. By turning in the test you confirm that **On your honor you have not received nor given any other student outside assistance**.

Make sure to show all your work. You may not receive full credit if the accompanying work is incomplete or incorrect. If you do scratch work make sure to indicate scratch work - I will not take off points for errors in the scratch work if it is so labeled. Make sure to distinguish between scratch work and proof.

Assume all spaces are topological Hausdorff spaces and, unless otherwise stated (as in problem 06), that $\mathbb{R}$ denotes the reals with the usual absolute value metric: $d(x, y) = |x - y|$ or (equivalently) the topology induced by the usual order $<$ with segments forming a basis.

In proving the theorems, you may use any of the theorems in our notes which precede (logically) the theorem to be proven. Make sure to provide as optimal a solution as possible. By this I mean that I am more interested in the clarity of presentation than the length. The problems in sections II and III are worth twice the points as those in section I. Section III is an experiment of sorts - we've been working on that theorem for a few weeks: I outlined my proof and I'd like students to provide the reasoning. Do as many claims as you can; I may weigh the first claims higher than the later ones.

Section I. Math 5500 students may omit one problem in this section (if all are attempted I will omit the one with the lowest score); Math 6500 students must do all the problems in this section.

Problem 01. Prove that $\overline{A} \cup \overline{B} = \overline{A \cup B}$.

Problem 02. Suppose that X is a metric space with metric d. Prove that if x is the sequential limit of the sequence $\{x_i\}_{i=1}^{\infty}$ and y is a point of X then $d(x, y)$ is the sequential limit of the sequence $\{d(x_i, y)\}_{i=1}^{\infty}$.

Problem 03a. Prove that a space is regular if and only if for each open set U and point $p \in U$ there is an open set V so that $p \in V \subset \overline{V} \subset U$.
Problem 03b. Prove that if each of X and Y is regular then $X \times Y$ is regular. [Hint: use part a.]

Problem 04. Prove that every Cauchy sequence in a compact metric space converges.

Problem 05. Consider the Reals $\mathbb{R}$ with the standard topology.

a.) Show that if $x \in \mathbb{R}$ then the singleton set $\{x\}$ is nowhere dense.

b.) Use Theorem 7.9 to argue that $\mathbb{R}$ is uncountable.

Problem 06. Consider the Reals $\mathbb{R}$, for $a, b \in \mathbb{R}$ define $(a, b] = \{x \in \mathbb{R} | a < x \leq b\}$ and define $\mathcal{B} = \{(a, b] | a, b \in \mathbb{R}\}$.

a.) Show that $\mathcal{B}$ satisfies the hypothesis of Theorem 2.2.

b.) Let $\mathcal{T}_{\mathcal{B}}$ be the topology on $\mathbb{R}$ generated by $\mathcal{B}$. Consider the following two sets:

$$M_1 = \left\{1 + \frac{1}{\sqrt{n}} \middle| n \in \mathbb{N}\right\} \qquad M_2 = \left\{1 - \frac{1}{\sqrt{n}} \middle| n \in \mathbb{N}\right\}.$$

For the space $(\mathbb{R}, \mathcal{T}_{\mathcal{B}})$, show that one of these sets has a limit point and the other does not.

c.) Show that every non-degenerate subset of $(\mathbb{R}, \mathcal{T}_{\mathcal{B}})$ is not connected. [Non-degenerate means having at least two elements.]

Problem 07. Prove that if X is a connected non-degenerate space then every point of X is a limit point of X.

Problem 08. Prove that a compact metric space is separable and completely separable.

Problem 09. Let $f : (0, 1) \to (1, \infty)$ be defined by:

$$f(x) = \frac{1}{1 - x}.$$

Show that f is a homeomorphism [one-to-one, onto, continuous and with a continuous inverse.]

Problem 10. If $f : X \to Y$ is a continuous onto function and X is separable, then Y is separable.

Problem 11. Suppose that X is a topological space and that the point p is a sequential limit of the sequence $\{x_i\}_{i=1}^{\infty}$ and of the sequence $\{y_i\}_{i=1}^{\infty}$. The sequence $\{z_i\}_{i=1}^{\infty}$ is formed as follows:

$$z_i = \begin{cases} x_{\frac{i}{2}} & \text{if } i \text{ is even} \\ y_{\frac{i+1}{2}} & \text{if } i \text{ is odd} \end{cases}$$

Prove that p is the sequential limit point of $\{z_i\}_{i=1}^{\infty}$.

Section II. Do two of A, B, C.

A.) Suppose that each of X and Y is compact and G is a covering of $X \times Y$ with basic open sets of the product space. You may use the facts proven in class that $X \times \{y\}$ is homeomorphic to X and $\{x\} \times Y$ is homeomorphic to Y in the following.

a.) Show that for each $x \in X$ that a finite subset of G covers $\{x\} \times Y$.

b.) Let $G' = \{g_i\}_{i=1}^{N}$ denote the finite collection found in part (a). Argue that $\cap_{i=1}^{N} \pi_1(g_i)$ is an open set that contains x.

c.) Prove that $X \times Y$ is compact.

B. Define a metric ρ on $\mathbb{R} \times \mathbb{R}$ as follows

$$\rho((a, b), (c, d)) \quad = \quad \max\{|a - c|, |b - d|\}.$$

a.) Show that ρ is a metric.

b.) Show that the metric ρ generates the product topology on $\mathbb{R} \times \mathbb{R}$

C. Prove that if S is a well ordered set and S is given the order topology (from the given well-ordering) then

a. The set of limit points S' of S is nowhere dense in S.

b. If $M \subset S$ is nowhere dense then $M \subset S'$.

Section III. Suppose that X is a complete metric space and d is the complete metric on X. Suppose that $\{M_i\}_{i=1}^{\infty}$ is a sequence of nowhere dense sets and that

$$X = \cup_{i=1}^{\infty} M_i.$$

We wish to arrive at a contradiction. We will use the fact that

$$\sum_{i=1}^{\infty} \frac{\delta}{2^i} = \delta.$$

Some notation. If $x \in X$ let $n(x)$ be the first integer n so that

$$x \in M_n.$$

Prove the following claims:

Claim 0. We can assume without lose of generality that each M_i is closed.

Claim 1. There is a point x_1 and a positive number δ_1 so that

$$x_1 \in X - M_1$$

and

$$\overline{B_{\delta_1}(x_1)} \subset X - M_1.$$

Claim 2. There is a point x_2 and a positive number $\delta_2 > 0$ so that:

$$\begin{aligned} x_2 &\in (X - \cup_{i=1}^{n(x_1)} M_i) \cap B_{\delta_1/2}(x_1) \\ \delta_1 &> \delta_2 \\ \overline{B_{\delta_2}(x_2)} &\subset (X - \cup_{i=1}^{n(x_1)} M_i) \cap B_{\delta_1/2}(x_1). \end{aligned}$$

Claim 3. There is a sequence of points $\{x_i\}_{i=1}^{\infty}$ and positive numbers $\{\delta_i\}_{i=1}^{\infty}$ (constructed inductively) so that:

$$\begin{aligned} x_{k+1} &\in (X - \cup_{i=1}^{n(x_k)} M_i) \cap B_{\delta_k/2}(x_k) \\ \delta_{k+1} &< \delta_k \\ \overline{B_{\delta_{k+1}}(x_{k+1})} &\subset (X - \cup_{i=1}^{n(x_k)} M_i) \cap B_{\delta_k/2}(x_k). \end{aligned}$$

Claim 4. The sequence $\{x_i\}_{i=1}^{\infty}$ is a Cauchy sequence. [Let x be the point to which the sequence converges.]

Claim 5. The sequence $\{x_i\}_{i=k}^{\infty}$ satisfies

$$\{x_i\}_{i=k}^{\infty} \subset B_{\delta_k}(x_k)$$

and so $\{x\} \cup \{x_i | i \geq k\}$ does not intersect $\cup_{i=1}^{n(k-1)} M_i$.

Claim 6. Therefore $x \notin M_i$ for all i which leads to a contradiction.