Topology notes.
Basic Definitions and Basic Properties.

Topology is the study of those properties of (geometric) objects (spaces) which are invariant under continuous and reversibly continuous transformations (i.e. continuous functions.) I assume that everyone has some familiarity with the concept of a function. So we need to define what it means for a function to be continuous. Continuity is based on the idea of an open set; in fact a *topology* on a space is the collection of open sets defined for the particular space under consideration.

Intuitively, a topological space X consists of a set of points and a collection $\mathcal{T}$ of special sets called open sets that provide information on how these points are related to each other. One can think of these points as a generalization of geometric points and the open sets as generalizations of geometric regions such as the inside of spheres or cubes. Thus for the definition of a topological space we are required to have a set of points for the underlying space and a collection of these open sets that define the "topology" on this space of points. We denote the topological space determined by the pointset X and topology $\mathcal{T}$ by $(X, \mathcal{T})$ though we will frequently refer to X as the topological space if the underlying topology is known.

Example 0.1. [The standard topology of the reals $\mathbb{R}$.] The set S is said to be a segment if and only if there are two numbers $a < b$ so that $S = \{x \mid a < x < b\}$; this segment S is denoted by (a, b). The set U is said to be open if and only if for each point $p \in U$ there is a segment S containing p so that $S \subset U$. Let $\mathcal{T}(\mathbb{R})$ be the collection of all open subsets of $\mathbb{R}$.

Example 0.1 B. [An alternate construction of the topology of the reals $\mathbb{R}$.] Let $\mathcal{B}$ be the set to which S belongs if and only if S is a segment so that there are two numbers rational numbers $a < b$ so that $S = (a, b) = \{x \mid a < x < b\}$. The set U is said to be open if and only if for each point $p \in U$ there is a segment $S \in \mathcal{B}$ containing p so that $S \subset U$. Let $\mathcal{T}_2(\mathbb{R})$ be the collection of all open subsets of $\mathbb{R}$.

Argue that $\mathcal{T}(\mathbb{R}) = \mathcal{T}_2(\mathbb{R})$.

Exercise 1.1. Define what it means for a set not to be open.

Exercise 1.2. Determine which of the following sets are open in $\mathbb{R}$:

a. A finite set.
b. The complement of a finite set.
c. $\{t|2t+1>5\}$.
d. $\{t|2t+1\geq 5\}$.
e. The integers.
f. The complement of the integers.
g. Rational numbers.
h. The set $\mathbb{R}$.
i. The empty set $\emptyset$.

Definition. If $S\subset\mathbb{R}$ then the *interior* of S, written $\mathrm{Int}(S)$ is the set to which p belongs if and only if there is a segment (a,b) so that $p\in(a,b)\subset S$.

Exercise 1.2B. Find the interior of the sets of exercise 1.2.

Exercise 1.2C. Find the interior of the following subsets of $\mathbb{R}$; determine in each case if the set is open.

j. $[1,3]$.
k. $[1,3)$.
l. $\{t|1-t^2\leq 5\}$.
m. $\cup_{n=1}^{\infty}(\frac{1}{2n+1},\frac{1}{2n})$.
n. $\cup_{n=1}^{\infty}[\frac{1}{2n+1},\frac{1}{2n}]$.
o. The complement of the set of (m).
p. The complement of the set of (n).
q. The irrational numbers.

Exercise 1.3. Show that the collection of open sets $\mathcal{T}(\mathbb{R})$ have the following properties:

1.) The sets $\mathbb{R}$ and $\emptyset$ are open.

2.) If A is open and B is open, then $A\cap B$ is open.

3.) If G is a collection of open sets, then the union of the elements of G is an open set. This set is denoted by $\cup G$ so that $\cup G=\{x\in\mathbb{R}|x\in g \text{ for some g } \in G\}$.

Exercise 1.4. In analysis a function $f:\mathbb{R}\to\mathbb{R}$ from the reals into the reals is said to be continuous at the point p if and only if: for each positive number ϵ there exists a positive number δ so that if:

$$|p - x| < \delta$$

then

$$|f(p) - f(x)| < \epsilon.$$

Show that this is equivalent to the statement that f is continuous at the point p if and only if for every open set V containing $f(p)$ there is an open set U containing p so that f maps every point of U into V.

Show that if "open set" in the above is replaced with "segment" then this produces another equivalent definition of continuity at the point p.

Exercise 1.4 b. [Continuation of exercise 1.4.] A function $f : \mathbb{R} \to \mathbb{R}$ is said to be continuous if and only if it is continuous at each of its points. Show that the function is continuous if and only if for each open set $U \in \mathcal{T}(\mathbb{R})$ the set $f^{-1}(U) = \{x|\ f(x) \in U\}$ is open.

Lemma (hint) for Exercise 1.4. The set U is open (in the reals $\mathbb{R}$) if and only if for each point $p \in U$ there exists a positive number $\epsilon > 0$ so that $(p - \epsilon, p + \epsilon) \subset U$.

Intuitively the boundary of an object is something right at the "edge" of the object and so is close to the object. Our intuition tells us that the boundary of the segment $S = (a, b)$ is the set of points $\{a, b\}$. An interval is a segment with these boundary points added. We use the notation $[a, b]$ to denote the interval $[a, b] = \{x \in \mathbb{R}|a \leq x \leq b\}$. Our goal in topology is to define concepts in terms of points and open sets.

Definition. If M is a set then the boundary of the set M, denoted by $Bd(M)$, is the set to which p belongs if and only if every open set containing p contains a point in M and a point not in M.

Exercise 1.5. Calculate the boundary of the sets from Exercise 1.2.

Exercise 1.6. Give an example of a set with the property that neither it nor its complement is open.

Theorem 1.1. If $U \subset \mathbb{R}$ is open, then U does not contain any of its boundary points.

Theorem 1.2. If $M \subset \mathbb{R}$ then $\mathbb{R} - Bd(M)$ is open.

Building on our analysis of the topology of the reals we now provide a formal definition of a topological space.

Definition. A *topological space* is a pair $(X, \mathcal{T})$ such that X is of a set of objects called *points* and $\mathcal{T}$ is a collection of subsets of X such that the following are satisfied:

1. $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$;
2. If $A \in \mathcal{T}$ and $B \in \mathcal{T}$ then $A \cap B \in \mathcal{T}$;
3. If $\mathcal{U} \subset \mathcal{T}$ then $\cup\{u | u \in \mathcal{U}\} \in \mathcal{T}$.

The elements of the collection $\mathcal{T}$ are called *open sets* and the collection $\mathcal{T}$ is called the *topology of* X.

If X and Y are two topological spaces, I may use the notation $\mathcal{T}(X)$ and $\mathcal{T}(Y)$ to denote their respective topologies in order to avoid confusion.

Since the topology of a space is dependent on the collection of open sets, the same underlying set of points may have different topologies. For any given set of points X we can consider the following two topologies.

Example 1.1. Let X be a set of points; let $\mathcal{T} = \{\emptyset, X\}$.

Example 1.2. Let X be a set of points; let $\mathcal{T} = \{H | H \subset X\}$.

Example 1.1 is often called the trivial or degenerate topology. Example 1.2 is called the discrete topology on a set. The first example describes the minimal topology based on the minimal requirements for a collection to be a topology on a space. The second is the largest topology possible for a set since it contains all the subsets of the space.

Exercise 1.7. Let X be a topological space. Suppose that for each positive integer i, U_i is an open set. Show that $\cup_{i=1}^{\infty} U_i$ is open and that if N is an integer then $\cap_{i=1}^{N} U_i$ is open.

Exercise 1.8. Show in the reals that there is a sequence of open sets $\{U_i\}_{i=1}^{\infty}$ so that $\cap_{i=1}^{\infty} U_i$ is not open.

Since the Euclidean plane can so easily be represented on the blackboard we define the standard topology for two dimensional Euclidean space $\mathbb{E}^2$. Let $X = \{(x, y) | \ x, \in \mathbb{R}\}$ denote the Euclidean plane. For each point $P = (a, b)$ and $\epsilon > 0$ let

$$\mathrm{B}_\epsilon(P) = \{(x, y) | \ \sqrt{(a - x)^2 + (b - y)^2} < \epsilon\}.$$

Then the set U is said to be open in X if and only if for each point $P \in U$ there is a positive number ϵ so that $\mathrm{B}_\epsilon(P) \subset U$. Let $\mathcal{T}_\circ$ denote this collection of open sets.

Exercise 1.9. Show that $(\mathbb{E}^2, \mathcal{T}_\circ)$ is a topological space.

Exercise 1.10. Let X be the Euclidian plane $\mathbb{E}^2$. For each point $P = (a, b)$ and $\epsilon > 0$ let

$$\hat{\mathrm{B}}_\epsilon(P) = \{(x, y) | \ |x - a| + |y - b| < \epsilon\}.$$

Then the set U is said to be open in X if and only if for each point $P \in U$ there is a positive number ϵ so that $\hat{\mathrm{B}}_\epsilon(P) \subset U$. Let $\mathcal{T}_\Diamond$ denote this collection of open sets.

Show that $(\mathbb{E}^2, \mathcal{T}_\Diamond)$ is a topological space.

Exercise 1.11. Show that $\mathcal{T}_\circ = \mathcal{T}_\Diamond$.

Now we give some common axioms that tell us something about the relationship of the points of a space and the open sets.

Axiom T_0. If p and q are points of X then there is an open set that contains one of these points and not the other.

Axiom T_1. If p and q are points of X then there is an open set that contains p and not q.

Axiom T_2. If p and q are points of X then there exist disjoint open sets A and B containing p and q respectively. A topological space that satisfies Axiom T_2 is called a Hausdorff space.

Exercise 1.12. Determine the implications among these three axioms. In other words determine if it is true that a topological space that satisfies

Axiom T_i also satisfies Axiom T_j. If an axiom T_i space does not satisfy Axiom T_j then give an example of a set of points X with a topology $\mathcal{T}$ so that $(X, \mathcal{T})$ satisfies Axiom T_i but does not satisfy T_j.

Exercise 1.13. Let X be a point set.
Let $\mathcal{T}_1 = \{\emptyset, X\}$.
Let $\mathcal{T}_2 = \{U | U \subset X\}$.
Determine whether or not these spaces are Hausdorff.

Unless otherwise stated, from this point on assume that all spaces are Hausdorff. Furthermore assume that X is always a topological space with a topology.

Definition. If $(X, \mathcal{T})$ is a topological space and $M \subset X$ is a point set then the point p is said to be a *limit point* of the set M if and only if each open set containing p contains a point of M distinct from p.

Theorem 1.3. If X is a finite Hausdorff space then no point of X is a limit point of X.

Exercise 1.14. Give an example of a finite set and three different topologies for that set. Can all three be Hausdorff?

Definition. Suppose $(X, \mathcal{T})$ is a topological space and $M \subset X$. Then the *derived set* of M denoted by M' is the set of limit points of M. The *closure* of M denoted by $\overline{M}$ is the set $M \cup M'$.

Theorem 1.4. If $(X, \mathcal{T})$ is a topological space and $M \subset X$ then $\overline{M} = \overline{\overline{M}}$.

Definition. The set $M \subset X$ is said to be *closed* if and only if every limit point of M is in M.

Theorem 1.5. Suppose $(X, \mathcal{T})$ is a topological space and $M \subset X$. Then M is closed if and only if $X - M$ is open.

Theorem 1.6. Suppose $(X, \mathcal{T})$ is a topological space and $M \subset X$. Then M' is closed.

Theorem 1.7. If p is a limit point of the set M, then every open set containing p contains infinitely many points of M.

Theorem 1.8. If each of A and B is a closed subset of the space X, then $A \cap B$ and $A \cup B$ are closed.

Question: Does Theorem 1.8 hold if X is not required to be Hausdorff? Is every space satisfying the conditions of Theorem 1.8 for arbitrary closed sets a Hausdorff space?

The definition of the boundary of a set given above in the case of the reals generalizes to arbitrary topological spaces. We state the definition for an arbitrary space.

Definition. Suppose $(X, \mathcal{T})$ is a topological space and $M \subset X$. Then the point p is called a *boundary point* of M iff every open set containing p contains a point in M and a point not in M. We denote the boundary of the set M by $Bd(M)$.

Exercise 1.15.

a. Find an example of a set that has no boundary point.

b. Find an example of a set every point of which is a boundary point.

[Note that as part of this exercise you need to define a space X and a topology on that space, and then find a set in that space that has the required property.]

Theorem 1.9. If X is a topological space and $M \subset X$, then $Bd(M) = \overline{M} \cap \overline{(X - M)}$.

Corollary 1.9. If X is a topological space and $M \subset X$, then $Bd(M)$ is closed.

Definition. Suppose $(X, \mathcal{T})$ is a topological space and $M \subset X$. Then the *interior* of M, denoted by $Int(M)$ is the set to which x belongs if and only if there is an open set containing x lying in M.

Exercise 1.16. In all the above theorems determine whether or not the theorem holds under the weaker T_0 or T_1 axioms.

Exercise 1.17. Suppose X is a topological space and A, B, M etc. are subsets of X. Prove the following. [In each case determine the weakest axiom that is needed to prove the statement. Caution: at least one of these is false! Note that I may not warn you in the future. In all the cases where the statement is false you should provide a counter example.]

a. $Int(M)$ is open.
b. $Int(Bd(M)) = \emptyset$.
c. $Int(Int(M)) = Int(M)$.
d. $Bd(Bd(M)) = Bd(M)$.
e. $Int(A \cap B) = Int(A) \cap Int(B)$.
f. $Bd(A \cap B) = Bd(A) \cap Bd(B)$.
g. $Int(M) \cap Int(X - M) = \emptyset$.
h. $(X - Int(M)) - Int(X - M) = Bd(M)$.
i. M is open iff $M \cap Bd(M) = \emptyset$.
j. $\overline{A \cup B} = \overline{A} \cup \overline{B}$.
k. $\overline{A \cap B} = \overline{A} \cap \overline{B}$.
l. U is open iff $U = Int(U)$.
m. $X - Int(A) = \overline{X - A}$.
n. $X = Int(M) \cup Bd(M) \cup Int(X - M)$.
o. A is closed iff $Bd(A) \subset A$.