Metric Spaces

Definition: Let X be a set and $\mathbb{R}$ the real numbers. The function $d : X \times X \to \mathbb{R}$ is called a *metric* for X provided:

1. $d(x, y) \geq 0$ for all $(x, y) \in X \times X$;
2. $d(x, y) = 0$ if and only if $x = y$;
3. $d(x, y) = d(y, x)$ for all $(x, y) \in X \times X$;
4. $d(x, z) \leq d(x, y) + d(y, z)$ for all x, y, z in X.

Definition: Suppose that X is a set, $x \in X$ and d is a metric for X. Then the set $B_\epsilon(x) = \{t \in X | \ d(t, x) < \epsilon\}$ is called the ϵ-ball around x.

Observation: you should notice that if X is a topological space and d is a metric for X. Then $\mathcal{B} = \{B_\epsilon(x) | x \in X, \epsilon > 0\}$ satisfies the hypothesis of theorem 2.2 and so is a basis for X.

Definition: (X, d) is said to be a metric space means that X is a topological space generated by the basis $\mathcal{B} = \{B_\epsilon(x) | x \in X, \epsilon > 0\}$.

Theorem 3.1. A metric space is Hausdorff.

Theorem 3.2. A metric space is first countable.

Theorem 3.3 A metric space is separable if and only it it is completely separable.

Definition. The topological space X is said to be *regular* iff for each point $x \in X$ and each closed subset $H \subset X$ of X not containing x, that there exist disjoint open sets U and V so that $x \in U$ and $H \subset V$.

Theorem 3.4. A metric space is regular.

Exercise 3.1. Show that the following "metrics" all produce the standard topology on $\mathbb{R}^2$.

a. $d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1, -y_2)^2}$, this is called the standard metric;

b. $d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1, -y_2|$, this is called the taxicab metric;

c. $d((x_1, y_1), (x_2, y_2)) = \min\{|x_1 - x_2|, |y_1, -y_2|\}$.

[Hint: In each case sketch a picture of the basic open sets.]

Exercise 3.2. Suppose that X is a space and d_1 and d_2 are two metrics for X. Then d_1 and d_2 produce the same topology if and only if for each point p and for positive number ϵ there are numbers r and s so that: $\{x|d_1(x, p) < r\} \subset \{x|d_2(x, p) < \epsilon\}$ and $\{x|d_2(x, p) < s\} \subset \{x|d_1(x, p) < \epsilon\}$.

Definition. The space X is said to me normal if and only if for each pair of disjoint closed sets H and K there exist disjoint open sets U and V so that $H \subset U$ and $K \subset V$.

Theorem 3.5. If X is a metric space then X is normal.

Theorem 3.6. Let X be a topological space.

a. Suppose that for each pair of points a and b that there exists a continuous function from X into $[0, 1]$ so that $f(a) = 0$ and $f(b) = 1$. Then X is Hausdorff.

b. Suppose that if a is a point and B is a closed set not containing a then there exists a continuous function from X into $[0, 1]$ so that $f(a) = 0$ and $f(B) = 1$. Then X is regular.

c. Suppose that for each pair of disjoint closed sets A and B that there exists a continuous function from X into $[0, 1]$ so that $f(A) = 0$ and $f(B) = 1$. Then X is normal.

[Notes. These (3.6 a-c) really are not hard once you look at them for awhile.

3.6 b. This is the definition of completely regular. To find a space that is completely regular but not regular is hard.

3.6 c. is actually an if and only if theorem; but the other direction is harder.

There is an example of a normal non-metric space.]

Theorem 3.7. Suppose that X and Y are topological spaces and there is a continuous, 1-1 and reversibly continuous onto function $f : X \to Y$. Then:

a. If X is Hausdorff, then so is Y,

b. If X is regular, then so is Y,

c. If X is normal, then so is Y.

Hand-in homework, due Friday September 27: Exercise 3.2 and prove Theorems 3.1, 3.2, 3.5, 3.6 and 3.7.

******************** Additional Exercises ***********************

Exercise: Let $X = \mathbb{R}$ and define a metric d on $\mathbb{R}$ by $d(x, y) = |y - x|$. Show that this metic induces the usual topology of the reals. As a hint to the proofs of the theorems above; prove them first for the space $\mathbb{R}$ with the usual topology.

Exercise: Let $X = \mathbb{R}$ and define a metric d on $\mathbb{R}$ by $d(x, y) = \min\{|y - x|, 1\}$. Show that this metric is equivalent to the one above. (I.e. generates the same topology.)

Exercise: Let $X = \mathbb{R} \times \mathbb{R}$. Define the following metric d on X:

$$d((x_1, y_1), (x_2, y_2)) = \begin{cases} \min\{|y_1 - y_2|, 1\} & \text{if } x_1 = x_2 \\ 1 & \text{if } x_1 \neq x_2. \end{cases}$$

Prove that d is in fact a metric and that the metric space produced is not separable.

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