Sequential Limit Points

Definition. The point p is the *sequential limit point* of the sequence $x_1, x_2, x_3, ...$ means that if U is an open set containing p then there exists an integer N so that if $n > N$ then $x_n \in U$.

Exercise 4.1. Consider the reals $\mathbb{R}$ with the standard topology. Show that:

a. The number 1 is the sequential limit of the sequence $\{\frac{n-1}{n+1}\}_{n=1}^{\infty}$.

b. The sequence $\{(-1)^n + \frac{1}{n}\}_{n=1}^{\infty}$ does not have a sequential limit point.

Theorem 4.1. If X is Hausdorff then the sequence $\{x_i\}_{i=1}^{\infty}$ has at most one sequential limit point.

Theorem 4.2. If X is Hausdorff and the point p is the sequential limit point of the sequence $\{x_i\}_{i=1}^{\infty}$ and the set $\{x_i | i \in \mathbb{Z}^+\}$ is finite, then there exists an integer N so that $x_n = p$ for all $n > N$.

Theorem 4.3. If X is first countable, $M \subset X$ and p is a limit point of M then there is an infinite sequence of distinct points of M $\{x_i\}_{i=1}^{\infty}$ so that p is a sequential limit point of $\{x_i\}_{i=1}^{\infty}$.

Exercise 4.2. Show in a metric space that x is the sequential limit point of the sequence $\{x_i\}_{i=1}^{\infty}$ if and only if for each $\epsilon > 0$ there exists an integer N_ϵ so that if $n > N_\epsilon$ then $d(x, x_n) < \epsilon$.

Theorem 4.4. Suppose that X and Y are Hausdorff spaces and $f : X \to Y$ is a continuous function and the point $p \in X$ is the sequential limit point of the sequence $\{x_i\}_{i=1}^{\infty}$. Then the point $f(p)$ is the sequential limit point of the sequence $\{f(x_i)\}_{i=1}^{\infty}$.