Compact Sets.

Definition. Suppose that X is a topological space, G is a collection of subsets of X and M is point set. Then the collection G is said to *cover* M if and only if each point in M lies in some element of G.

Definition. Suppose that X is a topological space. The subset M of X is said to be *compact* if and only if whenever G is a collection of open sets so that G covers M then a finite subcollection of G covers M.

Exercise 5.1. Show that the following subsets of the reals $\mathbb{R}$ with the usual topology is compact:

a. $M = \{x | x = 0 \text{ or } x = \frac{1}{n},\ n \in \mathbb{Z}^+\}$;

b. M is the unit interval $[0, 1]$. [Hint: you need to use the least upper bound axiom of the reals.]

Exercise 5.2. Show that the following subsets of the reals are not compact:

a. $M = \{x | x = \frac{1}{n},\ n \in Z^+\}$;

b. $M = \mathbb{R}$;

c. M is the open interval $(0, 1)$.

d. M is the set of positive integers.

e. $M = \mathbb{Q} \cap [0, 1]$.

Assume for the following theorems that all sets lie in a Hausdorff space X.

Theorem 5.1. If M is compact and H is a closed subset of M then H is compact.

Theorem 5.2. If M is compact then it is closed in X.

Theorem 5.3. If $f : X \to Y$ is continuous and $M \subset X$ is compact then $f(M)$ is compact.

Theorem 5.4. If M is compact and for each positive integer i, M_i is a non-empty closed subset of M so that $M_{i+1} \subset M_i$, then $\cap_{i=1}^{\infty} M_i \neq \emptyset$.

Theorem 5.5. Suppose that the space X is compact. Then it is regular.

Theorem 5.6. Suppose that the space X is compact. Then it is normal.

Definition. The collection G of subsets of X is said to be *monotonic* if and only if for each pair of sets H and K in G, one of them is a subset of the other.

Theorem 5.7. Suppose that M is compact and G is a monotonic collection of non-empty subsets of M. Then there is a point p so that p is a point or a limit point of every set in G.

Theorem 5.8. Suppose that X is a metric space, M is a compact subset of X and $p \in X$. Then there exists a closest point in M to p. [I.e. there is a point q in M so that no other point of M is closer to p than q.]

Theorem 5.9. Suppose that X is a metric space and that H and K are disjoint closed subsets of X and H is compact. Then there exists a positive number ϵ so that $d(h, k) > \epsilon$ for all $h \in H$ and $k \in K$.

Exercise 5.10. Show that theorem 5.9 is not true if the condition of compactness is removed.

Theorem 5.11. If M is compact then every infinite subset of M has a limit point.

Corollary to theorem 5.11 and exercise 5.1 b [Bolzano-Weierstrass theorem]: Every infinite and bounded subset of the reals has a limit point.