Ordered Topological Spaces.

Definition. Let S be a set. The relation $<$ is said to be an *order relation* on S iff:

1. If a and b are two points of S then either $a < b$ or $b < a$;
2. If $a < b$ and $b < c$, then $a < c$;
3. If $a < b$ then $b \not< a$;
4. For all $a \in S$, $a \not< a$.

Notation: If $<$ is an order relation on the set S then $a \leq b$ means $a < b$ or $a = b$. The relation $\geq$ is similarly defined.

Definition. Suppose that S is a set with the order relation $<$. If there is a point ℓ so that $x \leq \ell$ for all $x \in S$, then ℓ is called the last point of S.

If there is a point f so that $x \geq f$ for all $x \in S$ then f is called the first point of S.

Definition. Suppose that S is a set with an order relation $<$ then the order topology on S is constructed as follows:

For $a, b \in S$, define:

$$(a, b) \quad = \quad \{x | a < x < b\}.$$

Let $\mathcal{B}$ be the set to which B belong if and only if $B = (a, b)$ for some pair of elements a and b in S, $B = \{x \in S | x < a\}$ for some $a \in S$, or $B = \{x \in S | b < x\}$ for some $b \in S$.

Then $\mathcal{B}$ in a basis for the order topology.

Theorem 8.1. Suppose that S is a set with an order relation $<$ then the order topology on S is Hausdorff.

Definition. If S is a set with an order relation $<$ and $M \subset S$ then M *has a least element* means that there exists an element $p \in M$ so that $p \leq x$ for all $x \in M$.

Definition. Suppose that S is a set with an order relation $<$. Then the order relation $<$ is said to be a *well ordering* if and only if every subset of S has a least element.

Exercise 8.2. The positive integers is well ordered.

Exercise 8.3. The set of rational numbers with the usual ordering is not well ordered, but a (necessarily different) ordering can be defined on the rationals which is a well ordering.

Exercise 8.4. The set $\cup_{m=0}^{\infty}\{m+\frac{n-1}{n}|n \in \mathbb{Z}^{+}\}$ is well ordered by the usual ordering on the reals.

Theorem 8.5. Every subset of a well ordered set is well ordered.

Definition. If S is a set with the ordering $<$ and $M \subset S$, then b is a *lower bound* for M means $b \leq x$ for all $x \in M$; b is a *greatest lower bound* for M means that B is a lower bound for M and if $b' < b$ then b' is not a lower bound for M. S is said to have the *greatest lower bound property* iff whenever $M \subset S$ and M has a lower bound, then M has a greatest lower bound.

The concepts *upper bound*, *least upper bound* and least *upper bound property* are similarly defined.

Theorem 8.6. Let S be a well ordered set with the order topology. Then S has the greatest lower bound property.

Theorem 8.7. Let S be a well ordered set with the order topology. Then S has the least upper bound property.

Theorem 8.8. If S is an ordered set with a first and last element and it has the least upper bound property, then S with the order topology is compact.

Theorem 8.9. Let S be a well ordered set with the order topology which has a last element. Then S is compact.

Theorem 8.10. There is no infinite decreasing subset of a well ordered set.

Definitions for Exercise 8.10.

Suppose that A and B are two sets with order relations $<_A$ and $<_B$ respectively. Then A and B are said to be *order isomorphic* with respect to these ordering if and only if there is a 1-1 and onto function $f : A \to B$ so that $x <_A y$ if and only if $f(x) <_B f(y)$. If M is a set with order relation $<$ then I is said to be an *initial segment* of M if and only if $I \subset M$ and if $x \in I$ then $\{t \in M | \; t < x\} \subset I$.

Exercise 8.11. Let G denote the collection to which the subset g of the reals belongs if and only if g is well ordered with respect to the usual ordering on the reals. Define the relation "$\sim$" on G by $g_1 \sim g_2$ if and only if g_1 and g_2 are order isomorphic. Show that $\sim$ is an equivalence relation on G. Let $\mathcal{G}$ be the collection of equivalence classes of $\sim$; $\mathcal{G} = \{[g] | \; g \in G\}$. Define $[g_1] <_{\mathcal{G}} [g_2]$ if and only if $[g_1] \neq [g_2]$ and g_1 is order isomorphic to an initial segment of g_2. Show that:

1. $<_{\mathcal{G}}$ is an order relation on $\mathcal{G}$,
2. $<_{\mathcal{G}}$ is a well ordering,
3. $\mathcal{G}$ is uncountable,
4. Every initial segment of $\mathcal{G}$ is countable.

Theoretical stuff:

Axiom of choice. Suppose that $\mathcal{G}$ is a collection of sets; then there exists a function $F : \mathcal{G} \to \cup\mathcal{G}$ so that $F(g) \in g$ for every $g \in \mathcal{G}$.

Well ordering theorem. The Axiom of choice implies that every set can be well ordered.