Topology Notes 09
Product Spaces

Definition. Suppose that each of X and Y is a topological space. Then the topological cross product space denoted by $X \times Y$ is the space whose points are the point pairs: $X \times Y = \{(x, y)|x \in X, y \in Y\}$. A basis for the topology of $X \times Y$ is obtained as follows: if U is open in X and V is open in Y then $U \times V$ is a basis element.

Theorem 9.1. If each of X and Y is Hausdorff, then $X \times Y$ is Hausdorff.

Exercise. Give an example of an open set in a product space that is not a basis element as described above.

Exercise. Let Z be the product space $X \times Y$ and let $p \in X$. Show that the set $\{(p, y)|y \in Y\}$ is homeomorphic to Y.

Exercise. Let $Z = X \times X$. Show that the set $\{(x, x)|x \in X\}$ is homeomorphic to X.

Definition. If Z is the product space $X \times Y$ then the function $\pi_1 : Z \to X$ is defined by $\pi_1(x, y) = x$ and the function $\pi_2 : Z \to Y$ is defined by $\pi_2(x, y) = y$; these are called the projection maps.

Theorem 9.2. The projection maps π_1 and π_2 have the following properties.

a. They are continuous.
b. They map closed sets onto closed sets.
c. They map open sets onto open sets.

[Note one of these is deliberately false.]

Theorem 9.3. If S is a topological space and $Z = X \times Y$ then $f : S \to Z$ is continuous if and only if the functions $\pi_1 \circ f$ and $\pi_2 \circ f$ are continuous.

Theorem 9.4. If each of X and Y is a metric space then $X \times Y$ is a metric space.

Theorem 9.5. If each of X and Y is connected then $X \times Y$ is connected.

Theorem 9.6. If one of X and Y is not connected then $X \times Y$ is not connected.

Theorem 9.7. If each of X and Y is compact then $X \times Y$ is compact.