Test02, Math 5500, November 14, 2025
Dr. Michel Smith

Make sure to show all your work. You may not receive full credit if the accompanying work is incomplete or incorrect. If you do scratch work make sure to indicate scratch work - I will not take off points for errors in the scratch work if it is so labeled.

Note that all the proofs must follow logically from the theorems and definitions stated in the class notes; if you wish to use some lemma that has not been proven in class, you must prove it first using the theorems and definitions stated from the class notes.

Definitions [Use this page for definitions; I'll accept grammatically equivalent statements; 20 points]:

1. What does it mean to say that a set is connected.?

2. Define what it means for a set to be nowhere dense

3. Define what it means for a set to be well ordered.

4. Given topological spaces X_1 and X_2 with topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ respectively; define the topology of $X_1 \times X_2$.

Check the class notes for definitions.

Problems [80 points] Do four of the folllowing five problems; you may do all five for extra credit.

Problem 1. Let X be a non-degenerate connected topological space.

a.) Prove that every point of X is a limit point of X.

Proof. Let $x \in X$. Then $\{x\}$ and $X - \{x\}$ are disjoint sets whose union is X. Since X is nondegenerate, neither set is empty. Since X is connected one of the set must contain a limit point of the other. Since $\{x\}$ is closed then $X - \{x\}$ is open and so cannot contain a limit point of $\{x\}$ (or you may say, because $\{x\}$ is a single point $X - \{x\}$ cannot contain a limit point of $\{x\}$). So, $\{x\}$ contains a limit point of $X - \{x\})$, so x is a limit point of $X - \{x\}$, and so of X itself. □

b.) Prove that if H is a finite subset of X, then H is nowhere dense in X.

Proof. Since H is finite, H has no limit points and so is closed. Let U be an arbitrary nodegenerate open set. If $U \cap H = \emptyset$ then let $V = U$. Otherwise U contains a point x of H. By part (a) x is a limit point of X so U contains infinitely many points of $X - \{x\}$ and, since H is finite, one of them t is not in H so there is an open set W containing t and no point of H. In this case let $V = U \cap W$. In either case V is non-empty and lies in U. So, since U was arbitrary, H is nowhere dense in X. □

Problem 2. Let $\mathbb{R}$ denote the real numbers. Argue that $\mathbb{R}$ is uncountable.

[You may use the result of problem 1 to do this.]

Proof. Suppose that $\mathbb{R}$ is countable and that $\{x_n\}_{n=1}^{\infty}$ is an enumeration of the points of $\mathbb{R}$. By problem 1, $\{x_n\}$ is nowhere dense. So $\mathbb{R} = \cup_{n=1}^{\infty}\{x_n\}$. This produces $\mathbb{R}$ as the countable union of nowhere dense sets. But this contradicts the fact that $\mathbb{R}$ is a complete metric space (or a locally compact space). □

Problem 3. Show that every infinite well ordered set contains an order isomorphic copy of the positive integers.

[Hint: Use induction.]

Proof. Let S denote an infinite well-ordered set. We will construct an order preserving function f from the set $\{n\}_{n=1}^{\infty}$ to S. since S is well-ordered, every subset of S has a first element. Let s_1 denote the first element of S; define $f(1) = s_1$. Supose that we've constructed f to be order preserving from $\{1, 2, \ldots, n\}$ so that it's order preserving onto an initial segment of S of size n. Then let s_{n+1} be the first element of the set $S - \{f(1), f(2), \ldots, f(n)\}$ which is non-empty since S is infinite, and define $f(n + 1) = s_{n+1}$. Then by our choice of s_{n+1}, s_{n+1} follows every s_m for $m < n + 1$ and so f restricted to $\{1, 2, \ldots, n, n + 1\}$ is order preserving. So by the induction principle, f is an order preserving map from the positive integers into S.

□

Problem 4. Suppose that $[a, b]$ is an interval in $\mathbb{R}$ and $f : [a, b] \to \mathbb{R}$ is continuous.

a. Use the fact that $[a, b]$ is connected to argue that $f([a, b])$ is connected.

Proof. If $f([a, b])$ is not connected then $f([a, b])$ is the union of two disjoint compact sets (since we know that $f([a, b])$ is compact); C_1 and C_2. There exist disjoint open sets U_1 and U_2 containing C_1 and C_2 respectively. Then $f^{-1}(U_1)$ and $f^{-1}(U_2)$ are (non-empty) disjoint open sets whose union contains $[a, b]$ (and both intersect $[a, b]$) which contradicts the connectedness of $[a, b]$.

□

b. [The intermediate value theorem.] Prove that if d is between $f(a)$ and $f(b)$ then there is a point c with $a < c < b$ so that $f(c) = d$.

Proof. Let $C_1 = f^{-1}((-\infty, d))$ and $C_2 = f^{-1}((d, \infty))$. Then C_1 and C_2 are open sets and one contains $f(a)$ and the other contains $f(b)$ and so they have separated $f([a, b])$ by disjoint open sets which contradicts the connectedness of $f([a, b])$. □

Problem 5. Prove that the product of two compact spaces is compact.

Solution acceptable for a test. Let X and Y be compact and let $\mathcal{G}$ be a covering of $X \times Y$ by open sets. Then for each $p \in X \times Y$ let G_p be the element of $\mathcal{G}$ that contains p and then there exist open sets U_p of X and V_p of Y so that $p \in U_p \times V_p \subset G_p$. Note that $p = (x, y)$ with $x \in X$ and $y \in Y$. Since for each $x \in X$, $\{x\} \times Y$ is compact we can cover each $\{x\} \times Y$ with the open sets V_p's. For each x with $p = (x, y)$ the common part $\hat{U}_x$ of the corresponding U_x's is an open set containing x and the collection of all of them cover X. So a finite subcollection of the $\hat{U}_x$'s cover X. Then taking all the U_x that makeup these $\hat{U}_x$'s with each corresponding V_y gives you a finite covering in the form $U_x \times V_y$. So the G_p's that contain these open sets is the finite subcollection that we want. □

Proof with details. Let X and Y be compact and let $\mathcal{G}$ be a covering of $X \times Y$ by open sets. Then for each $p \in X \times Y$ let G_p be the element of $\mathcal{G}$ that contains p and then there exist open sets U_p of X and V_p of Y so that $p \in U_p \times V_p \subset G_p$. Note that $p = (x, y)$ with $x \in X$ and $y \in Y$. Since for each $x \in X$, $\{x\} \times Y$ is compact, there is a finite subset of $\{V_p | \pi_1(p) = x\}$ that covers $\{x\} \times Y$. Call this finite set $\{V_{(x,y(x,1))}, V_{(x,y(x,1))}, \ldots, V_{(x,y(x,n_x))}\}$; note that the numeber of terms $n(x)$ depend on x and the terms themselves depend on both coordinates of p hence the need for double subscripts (we use function notation: where $y(x, i)$ is the $i^{\text{th}} y$ associated with x). So for $p \in U_p \times V_p$ we have, $U_p = U_{(x,y(x,i))}$ and $V_p = V_{(x,y(x,i))}$ for some integer i. Define

$$\hat{U}_x = \bigcap_{i=1}^{n(x)} U_{(x,y(x,i))}.$$

Then $\hat{U}_x$ is an open set containing x so there is a finite subcollection of these covering X: $\hat{U}_{x_1}, \hat{U}_{x_2}, \ldots, \hat{U}_{x_N}$. Then the following collection covers $X \times Y$:

$$\left\{ U_{(x_i,y(x_i,k_i))} \times V_{(x_i,y(x_i,k_i)} \right\}_{k_i=1,i=1}^{n(x_i),N}.$$

And so the collection

$$\left\{ G_{(x_i,y(x_i,k_i)} \right\}_{k_i=1,i=1}^{n(x_i),N}.$$

is a finite subcollection of $\mathcal{G}$ that covers $X \times Y$. □