Test01, Math 5500, Qctober 3, 2025
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Make sure to show all your work. You may not receive full credit if the accompanying work is incomplete or incorrect. If you do scratch work make sure to indicate scratch work - I will not take off points for errors in the scratch work if it is so labeled.

Note that all the proofs must follow logically from the theorems and definitions stated in the class notes; if you wish to use some lemma that has not been proven in class, you must prove it first using the theorems and definitions stated from the class notes. If you are asked to "prove from the definition" then you must prove the statement from the definition without using any of the theorems.

Problems [80 points] Do four of the folllowing five problems; you may do all five for extra credit.

Problem 1. Suppose that X is a Hausdorff space and $M \subset X$:

a) Show that $\text{int}(M)$ contains no bounday points of M.

Proof. Let $p \in \text{int}(M)$ then, by definition, there exists an open set U contianing p that is a subset of M. Suppose p is also a boundary point of M then, by definition, U must contian a point of M and a point not in M. This is a contradiction, so p is not in the boundary of M.

□

b.) Show that $X = \text{int}(M) \cup \text{Bd}(M) \cup \text{int}(X - M)$.

Proof. Suppose $p \in X$ and p is in neither $\text{int}(M)$ nor in $\text{int}(X - M)$. Let U be an arbitrary open set containing p. Then, since $p \notin \text{int}(M)$, U must contain a point of $X - M$ and since $p \notin \text{int}(X - M)$, U must contain a point of M; but this is exactly the definition for a boundary point of M. So $X = \text{int}(M) \cup \text{Bd}(M) \cup \text{int}(X - M)$. □

Problem 2. Prove, from the definition, that a metric space satisfies the Hausdorff condition (axiom T_2).

Proof. Let x and y be two points of the metric space X amd let d be the metric on X. Then, $d(x,y) > 0$. Let $\epsilon = d(x,y)$ and consider the $\frac{\epsilon}{2}$ balls around x and y, $\mathrm{B}_{\epsilon/2}(x)$ and $\mathrm{B}_{\epsilon/2}(y)$ respectively. Let $U = \mathrm{B}_{\epsilon/2}(x)$ and $V = \mathrm{B}_{\epsilon/2}(y)$. Now assume that $U \cap V \neq \emptyset$ and $t \in U \cap V$, then

$$\begin{aligned} \epsilon &= d(x,y) \\ &\leq d(x,t) + d(y,t) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Which gives us $\epsilon < \epsilon$ which is a contradiction so $U \cap V = \emptyset$ with $x \in U$ and $y \in V$. This is the needed Hausorff condition. □

Problem 3. Prove that a closed subset of a compact space is compact.

Proof. Let M be a closed subset of the compact space X and let G be a covering of M with open sets. Since M is closed, for each $x \in X - M$, since M is closed, there exists an open set U_x contianing x and no point of M. Then the collection $G \cup \{U_x | x \in X - M\}$ is a collection of open sets covering X so some finite sub collection H also covers: $H \subset G \cup \{U_x | x \in X - M\}$. Let $H' = H - \{U_x | x \in X - M\}$; in other words, H' is the subcollection of H with all the elements from $\{U_x | x \in X - M\}$ that are in H but not in G removed. So $H' \subset G$ and since $H' \subset H$ it's a finite subcollection of G that covers M. Therfore, M is compact. □

Problem 4. Prove that a compact Hausdorff space is regular.

Proof. Let X be a Hausdorff space, let M be a closed subset of X and let p be a point of X not in M. Since X is Hausdorff, for each point $x \in M$ there exist disjoint open sets V_x and U_x containing p and x respectively. Since M is compact (by problem 3 or from class notes), there is a finite collection of $\{U_x | x \in X - M\}$ that covers M. Let $x_1, x_2, \ldots, x_n$ be the points so that $U_{x_1}, U_{x_2}, \ldots, U_{x_n}$ is the finite subcollection covering M. Then for

$$U = \cup_{i=1}^{n} U_{x_i}$$

and

$$V = \cap_{i=1}^{n} V_{x_i}$$

we have that U and V are open sets that are disjoint; and V and U contain p and M respectively. □

Problem 5. Suppose that X and Y are Hausdorff spaces and $f : X \to Y$ is a continuous onto function. Prove that if X is separable, then so is Y.

Proof. Let $\{r_i\}_{i=1}^{\infty}$ be the countable set that's dense in X (i.e. every open set of X contains some element of $\{r_i\}_{i=1}^{\infty}$). For each positive integer i let $y_i = f(r_i)$; we claim that $\{y_i\}_{i=1}^{\infty}$ is dense in Y. So, consider an open set U of Y. Since f is onto, U contains a point of the image $f(x)$ of X. Since f is continuous, $f^{-1}(U)$ is open (and non-empty) in X and so contains a point r_j in the dense set. So $y_j = f(r_j) \in U$. Therefore, since U was arbitrary, the set $\{y_i | i \in \mathbb{N}\}$ is countable and dense in Y. □