

The Well Ordering Theorem

Lemmas in the form of theorems and exercises

Definition. The relation $<$ is called an *order relation* means that it satisfies the following conditions:

If $x \neq y$ then either $x < y$ or $y < x$;

For all x , $x \not< x$;

If $x < y$ and $y < z$ then $x < z$.

Definition. Suppose that S is a set with an order relation $<$ and $M \subset S$. Then M has a least element means that there is an element $\ell \in M$ so that if $x \in M$ and $x \neq \ell$ then $\ell < x$.

Definition. Suppose that S is a set with an order relation $<$. Then S is said to be *well ordered* with respect to relation $<$ iff each subset of S has a least element.

Axiom of Choice: Suppose the G is a collection of sets. Then there is a function $F : G \rightarrow \cup G$ so that for each $g \in G$, $F(g) \in g$. The function F is called the choice function.

Well ordering theorem: The Axiom of Choice is equivalent to the statement that every set can be well ordered with respect to some ordering.

Exercise. Show the “easy” direction of the well ordering theorem.

For the following, assume the Axiom of Choice.

Definition. Suppose the S is an ordered set. Let B be the set to which b belongs if and only if:

there are two points x and y in S so that $b = \{t | x < t < y\}$,

there is a point x in S so that $b = \{t | x < t\}$, or

there is a point y in S so that $b = \{t | t < y\}$.

Then the *order topology* on S is the topology induced by letting B be a basis of open set.

To get a feeling of the property of well orderings, prove the following theorems.

Theorem. Suppose that the set S is well ordered with respect to the ordering $<$ and S has a last element. Then S with the order topology is compact.

Theorem. Suppose that the set S is well ordered with respect to the ordering $<$. Then there is no sequence $\{x_i\}_{i=1}^{\infty}$ so that for each i we have $x_{i+1} < x_i$.

Definition. Suppose that the set S is well ordered with respect to the ordering $<$. Then the subset I of S is called an *initial segment* if and only if either there is a point $x \in S$ so that $I = \{t | t < x\}$ or $I = S$. The initial segment I is called a proper initial segment of S if $I \neq S$.

Definition. Suppose that each of K and M is an ordered set with orders $<_K$ and $<_M$ respectively. Then the function $f : K \rightarrow M$ is called an *order isomorphism* if and only if for each pair of elements x and y in K so that $x <_K y$ we have $f(x) <_M f(y)$.

Theorem. Suppose that each of K and M is a well ordered set. Then either there is an order isomorphism from one of the sets onto the other or there is an order isomorphism from one of the sets onto an initial segment of the other.

Definition. The set X is said to be equally numerous with the set Y if and only if there is a one-to-one function of X onto Y .

Continuum Hypothesis [CH]: If X is an uncountable subset of the Reals $\mathbb{R}$ then X is equally numerous with $\mathbb{R}$.

Theorem [Assume CH and WOT]. There is a well ordering of the reals such that every proper initial segment is countable.

Now I'll present a list of lemmas that uses the Axiom of Choice to prove the well ordering theorem.

Suppose then that X is a set, $\mathcal{G}$ is the collection of all non-empty subsets of X and $F : \mathcal{G} \rightarrow X$ is a choice function so that

$$F(g) \in g.$$

Thus F selects an element of g for each $g \in \mathcal{G}$. We wish to use F to create a well ordering of the points of X . Before reading further ask yourself how you would use F to pick the element that will end up being the very first element of X . Then once you have selected the first element of X how would you select the second element of X .

!!Spoiler Alert!!

Make sure that you want to see the next page. I indicate how to find the first countable points of what will ultimately turn out to be our well-ordering.

I hope you suggested that the first element of X is the one selected from X by F . Thus the first element x_1 of X will be $F(X)$. This leads to the first lemma.

Lemma 1.

Let:

$$\begin{aligned} x_1 &= F(X), \\ x_2 &= F(X - \{x_1\}), \\ x_3 &= F(X - \{x_1, x_2\}), \\ &\vdots \\ x_{n+1} &= F(X - \{x_1, x_2, \dots, x_n\}), \\ &\vdots \\ x_{\omega_0} &= F(X - \{x_1, x_2, x_3, \dots\}); \end{aligned}$$

Define the order $<$ by $x_i < x_j$ if and only if $i < j$ or $j = \omega_0$.

Then the set $\{x_1, x_2, x_3, \dots, x_{\omega_0}\}$ is well ordered with respect to the ordering $<$.

We would like to extend this ordering “forever”. So first we need to show a certain kind of uniqueness.

Suppose that $g \in \mathcal{G}$ then we want to restrict ourselves to orderings of the type constructed in lemma 1. So let's define an ordering $<$ on the subset g of X to be *consistent with the choice function F* if and only if the first element of g is $x_1 = F(X)$ and that if I is an initial segment of g (with respect to the ordering on g) then $g - I$ has a first element and it is $F(X - I)$.

Lemma 2. Suppose that $g \in \mathcal{G}$ and $<$ is an ordering on the subset g of X that is consistent with the choice function F . Then $<$ is a well ordering.

Lemma 3. Suppose that $g \in \mathcal{G}$ and each of $<_1$ and $<_2$ is an ordering on g consistent with F . Then the two orderings are the same: for $x, y \in g$ we have $x <_1 y$ if and only if $x <_2 y$.

Lemma 4. Suppose that g_1 and g_2 are two subsets of X and that $<_1$ and $<_2$ are orderings of g_1 and g_2 respectively that are consistent with F . Then one of these sets is a subset of the other; furthermore, one is order isomorphic

to an initial segment of the other and the order isomorphism is the identity function. (I.e. one is literally an initial segment of the other.)

Lemma 5. Suppose that $\mathcal{H}$ is a collection of subsets of X so that for each $h \in \mathcal{H}$ there is an ordering $<_h$ on the subset h of X consistent with F and $H = \cup\{h|h \in \mathcal{H}\}$. Then the ordering $<$ defined by $x < y$ if and only if there exists $h \in \mathcal{H}$ containing both x and y so that $x <_h y$ is a well defined ordering on H that is consistent with F .

Lemma 6. Let $\mathcal{H}$ be the set to which h belongs if and only if $h \subset X$ and there exists an ordering on h consistent with F . Then $X = \cup\mathcal{H}$.

Conclusion. There is an ordering on X consistent with F and that ordering (by lemma 2) is a well ordering.

Second Spoiler Alert!

Now I want to construct an uncountable set and describe a specific well ordering on it. The next page gives a hint, then the next page after shows the construction.

The hint is to realize that every countable well ordered set can be embedded in the reals (in fact in an interval) in such a way that the ordering on the set matches the usual ordering on the reals. I.e. the set is order isomorphic to a subset of the reals with the usual ordering on the reals.

Before going on, try to find infinitely many different well ordered sets in the reals.

Try to find one with a limit point.

Then try to find one whose limit points has a limit point.

Then try to find one whose limit points of limit points has a limit point.

⋮

... and so on.

Let G denote the collection to which the subset g of the reals belongs if and only if g is well ordered with respect to the usual ordering on the reals. Define the relation “ $\sim$ ” on G by $g_1 \sim g_2$ if and only if g_1 and g_2 are order isomorphic. Show that $\sim$ is an equivalence relation on G . Let $\mathcal{G}$ be the collection of equivalence classes of $\sim$; $\mathcal{G} = \{[g] \mid g \in G\}$. Define $[g_1] <_{\mathcal{G}} [g_2]$ if and only if $[g_1] \neq [g_2]$ and g_1 is order isomorphic to an initial segment of g_2 . Show that:

1. $<_{\mathcal{G}}$ is an order relation on $\mathcal{G}$,
2. $<_{\mathcal{G}}$ is a well ordering,
3. $\mathcal{G}$ is uncountable,
4. Every initial segment of $\mathcal{G}$ is countable.

Zorn's lemma is another famous statement equivalent to the Axiom of Choice. I state here two versions of it. See if you can prove these without any lemmas.

Set version of Zorn's lemma.

Theorem [Axiom of Choice]. Suppose that G is a collection of sets that has the property that if G' is a monotonic subcollection of G then there is a set of the collection G which is a subset of every element of G' . Then there exists a set of the collection G which contains no other set of G .

Definition. The relation $<$ is called an *partial order relation* means that it satisfies the following conditions:

For all x , $x \not< x$;

If $x < y$ and $y < z$ then $x < z$.

Partial order version of Zorn's lemma.

Theorem [Axiom of Choice]. Suppose that M is a partially ordered set that has the property that if M' is a linearly ordered subset of M then there is an element of M that is less than or equal to every element of M' . Then there exists an element of M with no predecessor (such that no element of M precedes it).

Theorem. Zorn's lemma is equivalent to the Axiom of Choice.

Note that now that we have the well-ordering theorem, we can use it to prove Zorn's lemma.