Basis of a Topological Space

Definition. Suppose that $(X, \mathcal{T})$ is a topological space. Then $\mathcal{B} \subset \mathcal{T}$ is said to be a *basis* for the topology of X provided that for each point $p \in X$ and each open set $U \in \mathcal{T}$ containing p there is an element $B \in \mathcal{B}$ so that $p \in B \subset U$.

For the following assume that $(X, \mathcal{T})$ is a topological space and $\mathcal{B} \subset \mathcal{T}$ is a basis for the topology of X.

Theorem 2.1. The point p is a limit point of M iff every element of the basis $\mathcal{B}$ that contains p contains a point of M distinct from p.

Theorem 2.2. Suppose that $\mathcal{B}$ is a collection of subsets of X so that:

1. Every element of X is in some element of $\mathcal{B}$.
2. If $p \in X$ and A and B are elements of $\mathcal{B}$ both containing p, then there is an element of $\mathcal{B}$ containing p lying in $A \cap B$.

Then $\mathcal{T} = \{\cup W | W \subset \mathcal{B}\}$ is a topology for X. Definition: Under this hypothesis, the topology $\mathcal{T}$ is said to be generated by the basis $\mathcal{B}$.

Definition. Let $\mathbb{R}$ denote the reals. Let $\mathcal{B} = \{(a, b) = \{x | a < x < b\} | a \in \mathbb{R}, b \in \mathbb{R}\}$. Then the topology for $\mathbb{R}$ generated by $\mathcal{B}$ is called the standard topology for the reals $\mathbb{R}$.

Exercise 2.1. Let $\mathbb{E}^2 = \mathbb{R} \times \mathbb{R}$ denote the Euclidean plane. Let $\mathcal{B} = \{(a, b) \times (c, d) | a, b, c, d \in \mathbb{R}\}$. Then $\mathcal{B}$ generates the standard topology of $\mathbb{E}^2$.

Theorem 2.3. Suppose that $(X, \mathcal{T})$ is a topological space, $Y \subset X$, and $\mathcal{W} = \{U \cap Y | U \in \mathcal{T}\}$. Then $(Y, \mathcal{W})$ is a topological space.

Definition. If X and Y satisfy the hypothesis of Theorem 2.3 then Y is said to be a subspace of X with the subspace topology.

Corollary 2.4. If Y is a subspace of the topological space X and X is Hausdorff, then so is Y.

Exercise 2.2. Consider the standard topology of the reals. Show that p is a limit point of the set $M \subset \mathbb{R}$ iff for each positive number ϵ there exists a point x in M distinct from p so that $|p - x| < \epsilon$.

Exercise 2.3. Find an example of a topological space which has a basis of closed sets and such that every point of the space is a limit point of the space.

Definition. The function $f : X \to Y$ from the topological space X to the topological space Y is said to be *continuous at the point* $x \in X$ if for each open set $V \subset Y$ containing $f(x)$, there is an open set $U \subset X$ containing x so that every point of U is mapped into V by f.

Observation: The above definition is consistent with our definition of continuous. The function $f : X \to Y$ is continuous if it is continuous at each point in its domain.

Theorem 2.5. The function $f : \mathbb{R} \to \mathbb{R}$ is continuous iff $f(\overline{A}) \subset \overline{f(A)}$.

Theorem 2.6. If each of X, Y, and Z is a topological space and $f : X \to Y$ and $f : Y \to Z$ are continuous functions, then $g \circ f : X \to Z$ is continuous.

Definition. Let X and Y be topological spaces so that $f : X \to Y$ is a continuous 1-1 onto functions whose inverse is continuous. Then X and Y are said to be homeomorphic and f is called a homeomorphism.

Exercise 2.5. Determine which of the following properties are preserved under (1) onto continuous functions, (2) 1-1 and onto continuous functions, (3) homeomorphisms.

a. the space being Hausdorff;
b. sets being open;
c. sets being closed;
d. the boundary of sets;
e. the interior of sets;
f. points being limit points of sets.

Definition. If x is a point of the topological space X then the collection $\mathcal{B}_x \subset \mathcal{T}(X)$ is a *local basis* at x means that if U is an open set containing x then there is a member of $\mathcal{B}_x$ containing x and lying in U.

Definition. The space X is said to be *first countable* if there is a countable local basis at each of its points.

Definition. The space X is said to be *second countable* or *completely separable* if it has a countable basis.

Definition. The set $M \subset X$ is *dense* in X means that every non-empty open set contains an element of M. The space X is said to be *separable* if it contains a countable dense set.

Exercise 2.6. The reals with the standard topology is first countable, completely separable and separable.

Theorem 2.7. Suppose that X is a topological space.

a. If X is completely separable then it is separable.

b. If X is completely separable then it is first countable.

Theorem 2.8. If X is first countable at the point $x \in X$ then there exists a sequence of open sets $\{U_i\}_{i=1}^{\infty}$ so that $U_i \supset U_{i+1}$ for all positive integers i and $\{x\} = \cap_{i=1}^{\infty} U_i$.

Exercise 2.7. Determine which of the following properties are preserved under (1) onto continuous functions, (2) 1-1 and onto continuous functions, (3) homeomorphisms.

a. Separability.

b. Complete separability.

c. First countability.